The compactness and the concentration compactness via p-capacity

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Abstract

For $p \in (1, N)$ and $\Omega \subseteq \mathbb{R}^N$ open, the Beppo-Levi space $\mathcal{D}_0^{1,p}(\Omega)$ is the completion of $C_c^\infty(\Omega)$ with respect to the norm $\left[\int_\Omega |\nabla u|^p \, dx\right]^{\frac{1}{p}}$. Using the p-capacity, we define a norm and then identify the Banach function space $\mathcal{H}(\Omega)$ with the set of all g in $L^1_{loc}(\Omega)$ that admits the following Hardy-Sobolev type inequality:

$$\int_\Omega |g||u|^p \, dx \leq C \int_\Omega |\nabla u|^p \, dx, \forall \, u \in \mathcal{D}_0^{1,p}(\Omega),$$

for some $C > 0$. Further, we characterize the set of all g in $\mathcal{H}(\Omega)$ for which the map $G(u) = \displaystyle\int_\Omega g|u|^p \, dx$ is compact on $\mathcal{D}_0^{1,p}(\Omega)$. We use a variation of the concentration compactness lemma to give a sufficient condition on $g \in \mathcal{H}(\Omega)$ so that the best constant in the above inequality is attained in $\mathcal{D}_0^{1,p}(\Omega)$.

Mathematics Subject Classification (2020): 28A12, 28A33, 35A23, 35J20, 46E30, 46E35.

Keywords: Hardy-Sobolev inequality, concentration compactness, p-capacity, eigenvalue problem for p-Laplacian, absolute continuous norm, embedding of $\mathcal{D}_0^{1,p}(\Omega)$.

1. Introduction

For $p \in (1, N)$ and an open subset Ω of $\mathbb{R}^N$, the Beppo-Levi space $\mathcal{D}_0^{1,p}(\Omega)$ is the completion of $C_c^\infty(\Omega)$ with respect to the norm, $\|u\|_{\mathcal{D}} := \left[\int_\Omega |\nabla u|^p \, dx\right]^{\frac{1}{p}}$. We look for the weight function $g \in L^1_{loc}(\Omega)$ that admits the following Hardy-Sobolev type inequality:

$$\int_\Omega g|u|^p \, dx \leq C \int_\Omega |\nabla u|^p \, dx, \forall \, u \in \mathcal{D}_0^{1,p}(\Omega), \tag{1.1}$$

for some $C > 0$.

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Definition 1. *A function $g \in L^1_{loc}(\Omega)$ is called a Hardy potentials if $|g|$ satisfies (1.1). We denote the space of Hardy potentials by $\mathcal{H}(\Omega)$.*

Using Poincaré inequality, it is easy to verify that $L^\infty(\Omega) \subseteq \mathcal{H}(\Omega)$ if Ω is bounded (in one direction). Further, the classical Hardy-Sobolev inequality

$$\int_\Omega \frac{1}{|x|^p}|u|^p \, dx \leq \left(\frac{p}{N-p}\right)^p \int_\Omega |\nabla u|^p \, dx, \; u \in \mathcal{D}^{1,p}_0(\Omega) \tag{1.2}$$

ensures that $\frac{1}{|x|^p} \in \mathcal{H}(\Omega)$, even when Ω contains the origin. In the context of improving the Hardy-Sobolev inequality many examples of Hardy potentials were produced, see [12, 1, 17] and the references there in. For $p = 2$ and Ω bounded, $L^r(\Omega) \subseteq \mathcal{H}(\Omega)$ with $r > \frac{N}{2}$ [24], $r = \frac{N}{2}$ [2]. For $p \in (1, \infty)$ and for general domain Ω, in [30] authors showed that $L^{\frac{N}{p},\infty}(\Omega) \subseteq \mathcal{H}(\Omega)$ using the Lorentz-Sobolev embedding. If Ω is the exterior of closed unit ball, then examples of Hardy potentials outside the $L^{\frac{N}{p},\infty}(\Omega)$ are provided in [6]. For $g \in L^1_{loc}(\Omega)$, we consider

$$\tilde{g}(r) = \operatorname{ess\,sup}\{|g(y)| : |y| = r\}, \; r > 0,$$

where the essential supremum is taken with respect to $(N-1)$ dimensional surface measure. Let

$$I(\Omega) = \{g \in L^1_{loc}(\Omega) : \tilde{g} \in L^1((0,\infty), r^{p-1}\mathrm{d}r)\}; \qquad \|g\|_I = \int_0^\infty r^{p-1}|\tilde{g}|(r)dr.$$

Then, $I(\Omega)$ is a Banach space with the norm $\|.\|_I$ and $I(\Omega) \subseteq \mathcal{H}(\Omega)$ (Proposition 8).

In [25], Maz'ya gave a very intrinsic characterization of a Hardy potential using the p-capacity (see Section 2.4.1, page 128). Recall that, for $F \subset\subset \Omega$, the p-capacity of F relative to Ω is defined as,

$$\mathrm{Cap}_p(F, \Omega) = \inf\left\{\int_\Omega |\nabla u|^p \, dx : u \in \mathcal{N}(F)\right\},$$

where $\mathcal{N}(F) = \{u \in \mathcal{D}^{1,p}_0(\Omega) : u \geq 1 \text{ in a neighbourhood of } F\}$. Thus for $g \in \mathcal{H}(\Omega)$ and $w \in \mathcal{N}(F)$, we have

$$\int_F |g| \, dx \leq \int_\Omega |g||w|^p \, dx \leq C \int_\Omega |\nabla w|^p \, dx.$$

Now by taking the infimum over $\mathcal{N}(F)$ and as F is arbitrary, we get a necessary condition:

$$\sup_{F \subset\subset \Omega} \frac{\displaystyle\int_F |g| \, dx}{\mathrm{Cap}_p(F, \Omega)} \leq C.$$

Maz'ya proved that the above condition is also sufficient for g to be in $\mathcal{H}(\Omega)$. Motivated by this, for $g \in L^1_{loc}(\Omega)$, we define,

$$\|g\| = \sup\left\{ \frac{\displaystyle\int_F |g|\ dx}{\mathrm{Cap}_p(F,\Omega)} : F \subset\subset \Omega; |F| \neq 0 \right\}.$$

One can verify that $\|.\|$ is a Banach function norm on $\mathcal{H}(\Omega)$. The Banach function space structure of $\mathcal{H}(\Omega)$ and Maz'ya's characterization helps us to prove an embedding of $\mathcal{D}^{1,p}_0(\Omega)$ which is finer than the Lorentz-Sobolev embedding proved in [3]. We also provide an alternate proof for the Lorentz-Sobolev embedding (Proposition 10).

For $g \in \mathcal{H}(\Omega)$, let B_g be the best constant in (1.1). In this article, we are interested to find the Hardy potentials $g \in \mathcal{H}(\Omega)$ for which B_g is attained in $\mathcal{D}^{1,p}_0(\Omega)$. Many authors have considered similar problems in the context of finding the first (least) positive eigenvalue for the following weighted eigenvalue problem:

$$-\Delta_p u \ = \ \lambda g|u|^{p-2}u \ \text{ on } \mathcal{D}^{1,p}_0(\Omega). \tag{1.3}$$

If the map $G : \mathcal{D}^{1,p}_0(\Omega) \to \mathbb{R}$ defined by $G(u) = \displaystyle\int_\Omega g|u|^p\ dx$ is compact, then a direct variational method ensures that the first positive eigenvalue for the above problem exists and B_g is attained in $\mathcal{D}^{1,p}_0(\Omega)$. For $p = 2$ and Ω bounded, the compactness of G is proved for $g \in L^r(\Omega)$ with $r > \frac{N}{2}$ in [24] and $r = \frac{N}{2}$ in [2]. For $p \in (1,\infty)$ and for general domain Ω, $g \in L^{\frac{N}{p},d}(\Omega)$ with $d < \infty$, in [30]. The result is extended for a larger space $\mathcal{F}_{\frac{N}{p}}(\Omega) := \overline{C^\infty_c(\Omega)}$ in $L^{\frac{N}{p},\infty}(\Omega)$ in [7] for $p = 2$ and in [4] for $p \in (1,N)$. In [6], authors obtained the compactness of G for $g \in I(\overline{B}^c_1)$.

We extend and unify all the existing sufficient conditions for the compactness of G by characterizing the set of all Hardy potentials for which the map G is compact on $\mathcal{D}^{1,p}_0(\Omega)$. In fact, we provide three different characterizations and each of them uses the Banach function space structure of $\mathcal{H}(\Omega)$ in one way or other. Our first characterization is motivated by the definition of the space $\mathcal{F}_{\frac{N}{p}}(\Omega)$ considered in [7, 4]. Here, we consider the following subspace of $\mathcal{H}(\Omega)$:

$$\mathcal{F}(\Omega) := \overline{C^\infty_c(\Omega)} \text{ in } \mathcal{H}(\Omega).$$

Now, we have the following theorem:

Theorem 1. *Let $g \in \mathcal{H}(\Omega)$. Then $G : \mathcal{D}^{1,p}_0(\Omega) \to \mathbb{R}$ is compact if and only if $g \in \mathcal{F}(\Omega)$.*

For the second characterization, we use the notion of the absolute continuous norm on a Banach function space.

Definition 2. *Let $X = (X(\Omega), \|.\|_X)$ be a Banach function space. A function $f \in X$ is said to have absolute continuous norm, if for any sequence of measurable subsets (A_n) of Ω with χ_{A_n} converges to 0 a.e. on Ω, then $\|f\chi_{A_n}\|_X$ converges to 0.*

Theorem 2. *Let $g \in \mathcal{H}(\Omega)$. Then $G : \mathcal{D}_0^{1,p}(\Omega) \to \mathbb{R}$ is compact if and only if g has absolute continuous norm in $\mathcal{H}(\Omega)$.*

The third characterization is based on a concentration function that is defined using the norm on $\mathcal{H}(\Omega)$. For $x \in \overline{\Omega}$ and $r > 0$, let $B_r(x)$ be the ball of radius r centered at x. Now for $g \in \mathcal{H}(\Omega)$, we define,

$$\mathcal{C}_g(x) = \lim_{r\to 0}\left\|g\chi_{B_r(x)}\right\|, \qquad \mathcal{C}_g(\infty) = \lim_{R\to\infty}\left\|g\chi_{B_R(0)^c}\right\|.$$

Observe that, the concentration function $\mathcal{C}_g$ measures the lack of absolute continuity of the norm of g at all the points in Ω and at the infinity. Therefore, if $\mathcal{C}_g$ vanishes everywhere, then naturally one may anticipate the compactness of G, and precisely this is our next result.

Theorem 3. *Let $g \in \mathcal{H}(\Omega)$. Then $G : \mathcal{D}_0^{1,p}(\Omega) \to \mathbb{R}$ is compact if and only if*

$$\mathcal{C}_g(x) = 0, \ \forall x \in \overline{\Omega} \cup \{\infty\}.$$

Observe that the best constant in (1.1) is attained in $\mathcal{D}_0^{1,p}(\Omega)$ if and only if the following minimization problem (dual problem) has a minimizer:

$$\min\left\{\int_\Omega |\nabla u|^p \, dx : \ u \in \mathcal{D}_0^{1,p}(\Omega), \ \int_\Omega g|u|^p \, dx = 1\right\}. \tag{1.4}$$

If G is compact, then the level set $G^{-1}\{1\}$ is weakly closed and hence the weak limit of a minimizing sequence lie in $G^{-1}\{1\}$. Indeed, the weak limit of a minimizing sequence solves the minimization problem and B_g is attained at this weak limit. However, for the existence of the weak limit of a minimizing sequence, it is not necessary to have $G^{-1}\{1\}$ is weakly closed. In other words, for a non-compact G, (1.4) may admit a minimizer. These cases were treated in [29] for $p = 2$, $\Omega = \mathbb{R}^N$ and in [27] for $p \in (1, N)$ and general Ω. In [29]Smets and [27], authors provided sufficient condition on g for the existence of minimizer for (1.4). In [29], Tertikas used the celebrated concentration compactness lemma of Lions ([20, 21]) and Smets proved a variant of this lemma in [27]. One of their main restrictions was the countability of the closure of the 'singular' set of g (see Remark 5 for their definition of a singular set). In this article, we define the singular set as $\sum_g = \{x \in \overline{\Omega} : \mathcal{C}_g(x) > 0\}$ and in fact, $\sum_g$ coincides with the singular set considered by Tertikas [29] and Smets [27]. In the next theorem, we provide a sufficient condition which is weaker than the countability assumptions of [27, 29] for the existence of minimizer for (1.4).

Theorem 4. *Let $g \in \mathcal{H}(\Omega)$ be a non-negative function such that $\left|\overline{\sum_g}\right| = 0$ and*

$$C_H \mathcal{C}_g(x) < B_g, \forall x \in \overline{\Omega} \cup \{\infty\},$$

where $\left|\overline{\sum_g}\right|$ denotes the Lebesgue measure of $\overline{\sum_g}$, B_g is the best constant in (1.1) and $C_H = p^p(p-1)^{1-p}$. Then B_g is attained on $\mathcal{D}_0^{1,p}(\Omega)$.

If $g \in \mathcal{H}(\Omega)$ with $g \geq 0$, $\left|\overline{\sum_g}\right| = 0$ and $C_H dist(g, \mathcal{F}(\Omega)) < \|g\|$, then by the above theorem, B_g is attained on $\mathcal{D}_0^{1,p}(\Omega)$ (Corollary 2). This helps us to produce Hardy potentials for which the map G is not compact, however B_g is attained. The following theorem is an analogue of Theorem 1.3 of [29]:

Theorem 5. *Let $h \in \mathcal{H}(\Omega)$ with $h \geq 0$ and $\left|\overline{\sum_h}\right| = 0$. Then for any non-zero, non-negative $\phi \in \mathcal{F}(\Omega)$, there exists $\epsilon_0 > 0$ such that B_g is attained in $\mathcal{D}_0^{1,p}(\Omega)$ for $g = h + \epsilon\phi$, for all $\epsilon > \epsilon_0$.*

Remark 1. (i). We provide cylindrical Hardy potentials g for which $|\sum_g| = 0$, but $\sum_g$ is not countable (see Remark 14). Such cylindrical weights were considered by Badiale and Tarantello in [8] (for $N = 3$), Mancini et. al in [23] (for $N \geq 3$) to study certain semi-linear PDE involving Sobolev critical exponent. In astrophysics, such critical exponent problems with cylindrical weights often arises in the dynamics of galaxies [10, 13].
(ii). For a cylindrical Hardy potential $g \in \mathcal{H}(\Omega)$ with $|\overline{\sum_g}| = 0$, one can consider its perturbation $\tilde{g} := g + \phi$ by a suitable $\phi \in C_c^\infty(\Omega)$ and apply the above theorem to ensure $B_{\tilde{g}}$ is attained in $\mathcal{D}_0^{1,p}(\Omega)$ (see Remark 14 for a precise example). It is worth noticing that $|\overline{\sum_{\tilde{g}}}| = 0$ but not countable. Indeed, the results of [29, 27] are not applicable for such Hardy potentials.

The rest of the paper is organized as follows. In Section 2, we recall some important results that are required for the development of this article. Further, we discuss the function spaces $\mathcal{H}(\Omega)$, $\mathcal{F}(\Omega)$ and some embeddings of $\mathcal{D}_0^{1,p}(\Omega)$ in Section 3. In Section 4 we prove Theorem 1, Theorem 2 and Theorem 3. Section 5 contains the proof of Theorem 4 and Theorem 5.

2. Preliminaries

In this section, we briefly outline the symmetrization, Banach function space, Lorentz spaces and p-capacity and list some of their properties. Further, we state a few other results that we use in the subsequent sections.

2.1 Symmetrization

Let $\Omega \subseteq \mathbb{R}^N$ be an open set. Let $\mathcal{M}(\Omega)$ be the set of all extended real valued Lebesgue measurable functions that are finite a.e. in Ω. For $f \in \mathcal{M}(\Omega)$ and for $s > 0$, we define $E_f(s) = \{x : |f(x)| > s\}$ and the distribution function α_f of f is defined as

$$\alpha_f(s) \ := \ \big|E_f(s)\big|, \ \text{for } s > 0,$$

where $|A|$ denotes the Lebesgue measure of a set $A \subseteq \mathbb{R}^N$. Now, we define the *one dimensional decreasing rearrangement* f^* of f as below:

$$f^*(t) := \begin{cases} \text{ess sup} f, & t = 0 \\ \inf\{s > 0 \ : \ \alpha_f(s) < t\}, \ t > 0. \end{cases}$$

The map $f \mapsto f^*$ is not sub-additive. However, we obtain a sub-additive function from f^*, namely the maximal function f^{**} of f^*, defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(\tau)d\tau, \quad t > 0.$$

The sub-additivity of f^{**} with respect to f helps us to define norms in certain function spaces.

The *Schwarz symmetrization* of f is defined by

$$f^\star(x) = f^*(\omega_N|x|^N), \quad \forall\, x \in \Omega^\star,$$

where ω_N is the measure of the unit ball in $\mathbb{R}^N$ and $\Omega^\star$ is the open ball centered at the origin with same measure as Ω.

Next, we state an important inequality concerning the Schwarz symmetrization, see Theorem 3.2.10 of [14].

Proposition 1 (Hardy-Littlewood inequality)**.** *Let $\Omega \subseteq \mathbb{R}^N$ with $N \geq 1$ and $f, g \in \mathcal{M}(\Omega)$ be nonnegative functions. Then*

$$\int_\Omega f(x)g(x)\ dx \leq \int_{\Omega^\star} f^\star(x)g^\star(x)\ dx = \int_0^{|\Omega|} f^*(t)g^*(t)\ dt. \tag{2.1}$$

2.2 Banach function spaces

Definition 3. *A normed linear space $(X, \|.\|_X)$ of functions in $\mathcal{M}(\Omega)$ is called a Köthe function space if the following conditions are satisfied:*

1. $\|f\|_X = \|\ |f|\ \|_X$*, for all* $f \in X$*,*

2. *if* $g_1 \in X$ *and* $|g_2| \le |g_1|$ *a.e., then* $\|g_2\|_X \le \|g_1\|_X$ *and* $g_2 \in X$.

The norm $\|.\|_X$ is called a Köthe function norm on X. A complete Köthe function space $(X, \|.\|_X)$ is called as Banach function space and the associated norm $\|.\|_X$ is called a Banach function space norm.

Proposition 2. *[31, Theorem 2, Section 30, Chapter 6] Let* $(X, \|.\|_X)$ *be a Köthe function space such that, for any non-negative sequence of function* (f_n) *in* X *that increases to* f*, we have* $\|f_n\|_X$ *increases to* $\|f\|_X$*. Then* $(X, \|.\|_X)$ *is a Banach function space.*

For a Banach function space $(X, \|.\|_X)$, we define its associate space as follows.

Definition 4. *Let* $(X, \|.\|_X)$ *be a Banach function space. For* $u \in \mathcal{M}(\Omega)$*, define*

$$\|u\|_{X'} = \sup\left\{\int_\Omega |fu|\, dx : f \in X,\ \|f\|_X \le 1\right\}.$$

Then the associate space X' *of* X *is given by*

$$X' = \{u \in \mathcal{M}(\Omega) : \|u\|_{X'} < \infty\}.$$

One can verify that X' is also a Banach function space with respect to the norm $\|.\|_{X'}$. For further readings on Banach function spaces, we refer to [9, 14, 31].

2.3 Lorentz spaces

The Lorentz spaces are refinements of the usual Lebesgue spaces and introduced by Lorentz in [22]. For more details on Lorentz spaces and related results, we refer to the book [14].

Given a function $f \in \mathcal{M}(\Omega)$ and $(p, q) \in [1, \infty) \times [1, \infty]$ we consider the following quantity:

$$|f|_{(p,q)} := \left\| t^{\frac{1}{p}-\frac{1}{q}} f^*(t) \right\|_{L^q((0,\infty))} = \begin{cases} \left(\int_0^\infty \left[t^{\frac{1}{p}-\frac{1}{q}} f^*(t) \right]^q \, dt \right)^{\frac{1}{q}}; \ 1 \le q < \infty, \\ \sup\limits_{t>0} t^{\frac{1}{p}} f^*(t);\ q = \infty. \end{cases}$$

The Lorentz space $L^{p,q}(\Omega)$ is defined as

$$L^{p,q}(\Omega) := \left\{ f \in \mathcal{M}(\Omega) : |f|_{(p,q)} < \infty \right\}.$$

$|f|_{(p,q)}$ is a complete quasi-norm on $L^{p,q}(\Omega)$. For $(p, q) \in (1, \infty) \times [1, \infty]$, let

$$\|f\|_{(p,q)} := \left\| t^{\frac{1}{p}-\frac{1}{q}} f^{**}(t) \right\|_{L^q((0,\infty))}.$$

Then $\|f\|_{(p,q)}$ is a norm on $L^{p,q}(\Omega)$ and it is equivalent to the quasi-norm $|f|_{(p,q)}$ (see Lemma 3.4.6 of [14]). Next proposition identifies the associate space of Lorentz spaces, see [9] (Theorem 4.7, page 220).

Proposition 3. *Let $1 < p < \infty$ and $1 \leq q \leq \infty$ (or $p = q = 1$ or $p = q = \infty$). Then the associate space of $L^{p,q}(\Omega)$ is, up to equivalence of norms, the Lorentz space $L^{p',q'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.*

2.4 The p-capacity

For any subset A of $\mathbb{R}^N$ define,

$$\mathrm{Cap}_p(A) := \inf \left\{ \int_\Omega |\nabla u|^p \, dx : u \in \mathcal{D}_0^{1,p}(\mathbb{R}^N), A \subseteq \mathrm{int}\{u \geq 1\} \right\}.$$

It can be shown that the above definition is consistent with our earlier definition when A is relatively compact in Ω. Next, we list some of the properties of capacity in the following proposition.

Proposition 4. *(a) Let $\Omega_1 \subseteq \Omega_2$ be open in $\mathbb{R}^N$. Then $\mathrm{Cap}_p(., \Omega_2) \leq \mathrm{Cap}_p(., \Omega_1)$.*

(b) Cap_p is an outer measure on $\mathbb{R}^N$.

(c) For $\lambda > 0$ and $F \subset\subset \mathbb{R}^N$, $\mathrm{Cap}_p(\lambda F) = \lambda^{N-p}\mathrm{Cap}_p(F)$.

(d) For $F \subset\subset \mathbb{R}^N$, $\exists C > 0$ depending on p, N such that $|F| \leq C\mathrm{Cap}_p(F)^{\frac{N}{N-p}}$.

(e) For $N > p$, $\mathrm{Cap}_p(B_1) = N\omega_N \left(\frac{N-p}{p-1}\right)^{p-1}$, where B_1 is the unit ball in $\mathbb{R}^N$.

(f) $\mathrm{Cap}_p(L(F)) = \mathrm{Cap}_p(F)$, for any affine isometry $L : \mathbb{R}^N \mapsto \mathbb{R}^N$.

Proof. (a) Follows easily from the definition of capacity.
(b) See Theorem 4.14 of [15](page 174).
$(c), (d), (f)$ See Theorem 4.15 of [15](page 175).
(e) Section 2.2.4 of [25] (page 106). ☐

Remark 2. If a set A is measurable with respect to Cap_p then $\mathrm{Cap}_p(A)$ must be 0 or ∞, see Theorem 4.14 of [15] (page 174).

The next theorem follows from Maz'ya's characterization of Hardy potential, [25] (see Section 2.3.2, page 111).

Theorem 6. *Let $p \in (1, N)$, $\Omega \subseteq \mathbb{R}^N$ be open and $g \in L^1_{loc}(\Omega)$. If $g \in \mathcal{H}(\Omega)$, then*

$$\int_\Omega g|u|^p \, dx \leq C_H \|g\| \int_\Omega |\nabla u|^p \, dx, \forall u \in \mathcal{D}^{1,p}_0(\Omega),$$

where $C_H = p^p(p-1)^{1-p}$.

It is easy to see that, the best constant satisfies the following inequalities:

$$\|g\| \leq B_g \leq C_H \|g\|. \tag{2.2}$$

2.5 The space of measures

Let $\mathbb{M}(\mathbb{R}^N)$ be the space of all bounded signed measures on $\mathbb{R}^N$. Then $\mathbb{M}(\mathbb{R}^N)$ is a Banach space with respect to the norm $\|\mu\| = |\mu|(\mathbb{R}^N)$ (total variation of the measure μ). The next proposition follows from the uniqueness of the Riesz representation theorem.

Proposition 5. *Let $\mu \in \mathbb{M}(\mathbb{R}^N)$ be a positive measure. Then for an open $V \subseteq \mathbb{R}^N$,*

$$\mu(V) = \sup \left\{ \int_{\mathbb{R}^N} \phi \, d\mu : 0 \leq \phi \leq 1, \phi \in C_c^\infty(\mathbb{R}^N) \text{ with } Supp(\phi) \subseteq V \right\}$$

and for any Borel set $E \subseteq \Omega$, $\mu(E) := \inf \{\mu(V) : E \subseteq V, V \, open\}$.

Recall that, a sequence (μ_n) is said to be weak* convergent to μ in $\mathbb{M}(\mathbb{R}^N)$, if

$$\int_{\mathbb{R}^N} \phi \, d\mu_n \to \int_{\mathbb{R}^N} \phi \, d\mu, \text{ as } n \to \infty, \forall \, \phi \in C_c(\mathbb{R}^N).$$

In this case we denote $\mu_n \overset{*}{\rightharpoonup} \mu$. The next proposition is a consequence of Banach-Alaoglu theorem which states that for any Banach space X, the closed unit ball in X^* is weak* compact.

Proposition 6. *Let (μ_n) be a bounded sequence in $\mathbb{M}(\mathbb{R}^N)$, then there exists $\mu \in \mathbb{M}(\mathbb{R}^N)$ such that $\mu_n \overset{*}{\rightharpoonup} \mu$ up to a subsequence.*

2.6 Brezis-Lieb lemma

The following lemma is due to Brezis and Lieb (see Theorem 1 of [11]).

Lemma 1. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and (f_n) be a sequence of complex -valued measurable functions which are uniformly bounded in $L^p(\Omega, \mu)$ for some $0 < p < \infty$. Moreover, if (f_n) converges to f a.e., then*

$$\lim_{n\to\infty} \left| \|f_n\|_{(p,\mu)} - \|f_n - f\|_{(p,\mu)} \right| = \|f\|_{(p,\mu)}.$$

We also require the following inequality (see [19], page 22) that played an important role in the proof of Bresiz-Lieb lemma: for $a, b \in \mathbb{C}$,

$$\big||a+b|^p - |a|^p\big| \leq \epsilon |a|^p + C(\epsilon, p)|b|^p \tag{2.3}$$

valid for each $\epsilon > 0$ and $0 < p < \infty$.

3. Embeddings

In this section we prove the following continuous embeddings:

$$L^{\frac{N}{p},\infty}(\Omega) \hookrightarrow \mathcal{H}(\Omega); \quad \mathcal{F}_{\frac{N}{p}}(\Omega) \hookrightarrow \mathcal{F}(\Omega); \quad I(\Omega) \hookrightarrow \mathcal{F}(\Omega).$$

We provide alternate proofs for certain classical embeddings and also provide an embedding of $\mathcal{D}_0^{1,p}(\Omega)$ finer than Lorentz-Sobolev embeddings.

Proposition 7. *For $p \in (1, N)$ and an open subset Ω in $\mathbb{R}^N$, $L^{\frac{N}{p},\infty}(\Omega)$ is continuously embedded in $\mathcal{H}(\Omega)$.*

Proof. Observe that, $\mathrm{Cap}_p(F^\star) \leq \mathrm{Cap}_p(F^\star, \Omega^\star) \leq \mathrm{Cap}_p(F, \Omega)$. The first inequality comes from (a)-th property of Proposition 4 and the latter one follows from Polya-Szego inequality. $\mathrm{Cap}_p(F^\star) = N\omega_N(\frac{N-p}{p-1})^{p-1}R^{N-p}$, where R is the radius of $F^\star$ (by (e)-th property of Proposition 4). Now, for a relatively compact set F,

$$\frac{\int_F |g|(x)\, dx}{\mathrm{Cap}_p(F, \Omega)} \leq \frac{\int_{F^\star} g^\star(x)\, dx}{\mathrm{Cap}_p(F^\star, \mathbb{R}^N)} = \frac{\int_0^{|F|} g^*(t)\, dt}{N\omega_N(\frac{N-p}{p-1})^{p-1}R^{N-p}} = \frac{R^p g^{**}(\omega_N R^N)}{N(\frac{N-p}{p-1})^{p-1}}.$$

By setting $\omega_N R^N = t$ we get,

$$\frac{\int_F |g|(x)\, dx}{\mathrm{Cap}_p(F, \Omega)} \leq C(N,p)\|g\|_{(\frac{N}{p},\infty)}.$$

Now take the supremum over $F \subset\subset \Omega$ to obtain,

$$\|g\| \leq C(N,p)\|g\|_{(\frac{N}{p},\infty)} \text{ with } C(N,p) = \frac{1}{N(\omega_N)^{\frac{p}{N}}(\frac{N-p}{p-1})^{p-1}}.$$

□

Proposition 8. *For $p \in (1, N)$ and an open subset Ω in $\mathbb{R}^N$, $I(\Omega)$ is continuously embedded into $\mathcal{H}(\Omega)$.*

Proof. For $g \in I(\Omega)$ and $u \in \mathcal{N}(F)$, use Lemma 2.1 of [6] to obtain

$$\int_F |g| \, dx \le \int_\Omega |g||u|^p dx \le C_H \|g\|_I \int_\Omega |\nabla u|^p dx, \ \forall u \in \mathcal{D}_0^{1,p}(\Omega),$$

where C depends only on N, p. Taking the infimum over $\mathcal{N}(F)$ and then the supremum over F we obtain $\|g\| \le C_H \|g\|_I$. $\square$

Remark 3. Notice that $I(\Omega)$ and $\mathcal{H}(\Omega)$ are not rearrangement invariant Banach function spaces. For example, for $p = 2$, $N \ge 3$ and $\beta \in (\frac{2}{N}, 1)$ consider the following function analogous to Example 3.8 of [5],

$$g(x) = \begin{cases} (|x| - 1)^{-\beta}, & 1 < |x| \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

It can be verified that $g \in I(\mathbb{R}^N)$ and $g^\star \notin \mathcal{H}(\mathbb{R}^N)$.

As we have mentioned before, if $g \in \mathcal{F}_{\frac{N}{p}}(\Omega)$ then G is compact and the same is true if $g \in I(\Omega)$, where $\Omega = \overline{B_1}^c$. Next proposition shows that $\mathcal{F}(\Omega)$ contains these spaces.

Proposition 9. *Let $p \in (1, N)$. Then*

(i) $\mathcal{F}_{\frac{N}{p}}(\Omega) \subseteq \mathcal{F}(\Omega)$ for any open subset Ω in $\mathbb{R}^N$.

(ii) $I(\Omega) \subseteq \mathcal{F}(\Omega)$ for $\Omega = B_d \setminus \overline{B_c}$; $0 \le c < d \le \infty$.

Proof. Recall that, $\mathcal{F}_{\frac{N}{p}}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $L^{\frac{N}{p},\infty}(\Omega)$ and $\mathcal{F}(\Omega)$ is closure of $C_c^\infty(\Omega)$ in $\mathcal{H}(\Omega)$. Now since $\|.\| \le C\|.\|_{(\frac{N}{p},\infty)}$, it is immediate that $\mathcal{F}_{\frac{N}{p}}(\Omega)$ is contained in $\mathcal{F}(\Omega)$. Similarly, in order to prove (ii), it is enough to show $C_c^\infty(\Omega)$ is dense in $I(\Omega)$. For this, let $g \in I(\Omega)$ and $\epsilon > 0$ be arbitrary. As $C_c^\infty((c,d))$ is dense in $L^1((c,d), r^{p-1})$, there exists $\phi \in C_c^\infty((c,d))$ such that $\|\tilde{g} - \phi\|_{L^1((c,d),r^{p-1})} < \epsilon$. Now, for $x \in \Omega$ let $\sigma(x) := \phi(|x|)$. By denoting, $h = g - \sigma$ we have, $\tilde{h}(r) = \tilde{g}(r) - \phi(r)$. Therefore,

$$\|g - \sigma\|_I = \int_0^\infty |\tilde{h}|(r) r^{p-1} dr = \|\tilde{g} - \phi\|_{L^1((0,\infty),r^{p-1})} < \epsilon.$$

$\square$

Remark 4. In [4, Lemma 3.5], authors have shown that $\mathcal{F}_{\frac{N}{p}}(\Omega)$ contains the Hardy potentials that have faster decay than $\frac{1}{|x-a|^p}$ at all points $a \in \overline{\Omega}$ and at infinity. Such Hardy potentials arises in the work of Szulkin and Willem [28]. Above proposition assures that they belong to $\mathcal{F}(\Omega)$.

Next, we give an alternate proof for the Lorentz-Sobolev embedding of $\mathcal{D}_0^{1,p}(\Omega)$. The idea is similar to that of Corollary 3.6 of [5].

Proposition 10. *For $p \in (1,N)$ and an open subset Ω in $\mathbb{R}^N$, $\mathcal{D}_0^{1,p}(\Omega)$ is continuously embedded in $L^{p^*,p}(\Omega)$.*

Proof. Without loss of generality we may assume $\Omega = \mathbb{R}^N$ (for a general domain Ω, the result will follow by considering the zero extension to $\mathbb{R}^N$). Let $g \in \mathcal{M}(\Omega)$ be such that $g^\star \in \mathcal{M}(\Omega)$. Then using the Polya-Szego inequality we have,

$$\int_{\mathbb{R}^N} g^\star |u^\star|^p \, dx \leq C_H \|g^\star\| \int_{\mathbb{R}^N} |\nabla u^\star|^p \, dx \leq C_H \|g^\star\| \int_{\mathbb{R}^N} |\nabla u|^p \, dx, \ \forall u \in \mathcal{D}_0^{1,p}(\mathbb{R}^N).$$

In particular, for $g(x) = \frac{1}{\omega_N^{\frac{p}{N}} |x|^p}$, $g^*(s) = \frac{1}{s^{\frac{p}{N}}}$ and $\|g^\star\| = \frac{(p-1)^{p-1}}{N(N-p)^{p-1}}$. Now $\int_{\mathbb{R}^N} g^\star |u^\star|^p \, dx = \int_0^\infty g^*(s) |u^*(s)|^p \, ds$. Thus from the above inequality we obtain,

$$\int_0^\infty \frac{|u^*(s)|^p}{s^{\frac{p}{N}}} \, ds \leq C(N,p) \int_{\mathbb{R}^N} |\nabla u|^p \, dx, \, \forall \, u \in \mathcal{D}_0^{1,p}(\Omega).$$

The left hand side of the above inequality is $|u|^p_{(p^*,p)}$, a quasi-norm equivalent to the norm $\|u\|^p_{(p^*,p)}$ in $L^{p^*,p}(\Omega)$. This completes the proof. □

Corollary 1. *For $p \in (1,N)$ and an open subset Ω in $\mathbb{R}^N$, $\mathcal{D}_0^{1,p}(\Omega)$ is compactly embedded in $L^p_{loc}(\Omega)$.*

Proof. Clearly $\mathcal{D}_0^{1,p}(\Omega)$ is continuously embedded into $W^{1,p}_{loc}(\Omega)$. Since $W^{1,p}_{loc}(\Omega)$ is compactly embedded in $L^p_{loc}(\Omega)$, we have the required embedding. □

Notice that we used just one Hardy potential $\frac{1}{|x|^p}$ to obtain the Lorentz-Sobolev embedding in Proposition 10. Instead, if we consider all of $\mathcal{H}(\Omega)$, then we anticipate to get an embedding finer than the above one. For this, we consider the following space (defined in a similar way as the associate space):

$$\mathcal{E}(\Omega) := \left\{ u \in \mathcal{M}(\Omega) : |u|^p \in \mathcal{H}(\Omega)' \right\}.$$

One can verify that $\mathcal{E}(\Omega)$ is a Banach function space with respect to the norm

$$\|u\|_{\mathcal{E}} := \left(\| |u|^p \|' \right)^{\frac{1}{p}}.$$

In the next theorem, we establish an embedding of $\mathcal{D}_0^{1,p}(\Omega)$ into $\mathcal{E}(\Omega)$. Further, we assert that the embedding is finer than the classical one.

Theorem 7. *Let $1 < p < N$ and Ω be open in $\mathbb{R}^N$. Then*

(a) $\mathcal{D}_0^{1,p}(\Omega)$ is continuously embedded into $\mathcal{E}(\Omega)$,

(b) $\mathcal{E}(\Omega)$ is a proper subspace of $L^{p^*,p}(\Omega)$.

Proof. (a) For $g \in \mathcal{H}(\Omega)$, by Theorem 6,

$$\int_\Omega g|u|^p \, dx \leq C_H \|g\| \int_\Omega |\nabla u|^p \, dx, \forall u \in \mathcal{D}_0^{1,p}(\Omega).$$

Now taking the supremum over the unit ball in $\mathcal{H}(\Omega)$ we obtain,

$$\|u\|_{\mathcal{E}} \leq C_H^{\frac{1}{p}} \|u\|_{\mathcal{D}_0^{1,p}(\Omega)}, \forall u \in \mathcal{D}_0^{1,p}(\Omega).$$

(b) Clearly $v \in \mathcal{E}(\Omega)$ if and only if $|v|^p \in \mathcal{H}(\Omega)'$. Further, $L^{\frac{N}{p},\infty}(\Omega) \subsetneq \mathcal{H}(\Omega)$ and hence $\mathcal{H}(\Omega)' \subsetneq L^{\frac{p^*}{p},1}(\Omega)$ (by Proposition 3). Now, we can easily deduce that $\mathcal{E}(\Omega) \subsetneq L^{p^*,p}(\Omega)$.

□

4. The compactness

In this section, we develop a g depended concentration compactness lemma as in [27]. Then we give equivalent conditions for compactness and prove Theorem 1, Theorem 2 and Theorem 3.

Lemma 2. *Let $\Phi \in C_b^1(\Omega)$ be such that $\nabla\Phi$ has compact support and $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$. Then*

$$\overline{\lim_{n\to\infty}} \int_\Omega |\nabla((u_n - u)\Phi)|^p \, dx = \overline{\lim_{n\to\infty}} \int_\Omega |\nabla(u_n - u)|^p |\Phi|^p \, dx.$$

Proof. Let $\epsilon > 0$ be given. Using (2.3),

$$\left| \int_\Omega |\nabla((u_n - u)\Phi)|^p \, dx - \int_\Omega |\nabla(u_n - u)|^p |\Phi|^p \, dx \right| \leq \epsilon \int_\Omega |\nabla(u_n - u)|^p |\Phi|^p \, dx + C(\epsilon, p) \int_\Omega |u_n - u|^p |\nabla\Phi|^p \, dx.$$

Since $\nabla\Phi$ is compactly supported, by Corollary 1 the second term in the right-hand side of the above inequality goes to 0 as $n \to \infty$. Further, as (u_n) is bounded in $\mathcal{D}_0^{1,p}(\Omega)$ and $\epsilon > 0$ is arbitrary, we obtain the desired result. □

A function in $\mathcal{D}_0^{1,p}(\Omega)$ can be considered as a function in $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$ by usual zero extension. Following this convention, for $u_n, u \in \mathcal{D}_0^{1,p}(\Omega)$ and a Borel set E in $\mathbb{R}^N$, we denote

$$\nu_n(E) = \int_E g|u_n - u|^p \, dx; \qquad \Gamma_n(E) = \int_E |\nabla(u_n - u)|^p \, dx; \qquad \widetilde{\Gamma}_n(E) = \int_E |\nabla u_n|^p \, dx.$$

If $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$, then ν_n, Γ_n and $\widetilde{\Gamma}_n$ have weak* convergent sub-sequences (Proposition 6). Let

$$\nu_n \overset{*}{\rightharpoonup} \nu; \qquad \Gamma_n \overset{*}{\rightharpoonup} \Gamma; \qquad \widetilde{\Gamma}_n \overset{*}{\rightharpoonup} \widetilde{\Gamma} \text{ in } \mathbb{M}(\mathbb{R}^N).$$

Next, we prove the absolute continuity of the measure ν with respect to Γ.

Lemma 3. *Let $g \in \mathcal{H}(\Omega)$, $g \geq 0$ and $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$. Then for any Borel set E in $\mathbb{R}^N$,*

$$\nu(E) \leq C_H \; \mathcal{C}_g^* \Gamma(E), \text{ where } \mathcal{C}_g^* = \sup_{x \in \overline{\Omega}} \mathcal{C}_g(x) \; .$$

Proof. As $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$, $u_n \to u$ in $L^p_{loc}(\Omega)$ (by Corollary 1). For $\Phi \in C_c^\infty(\mathbb{R}^N)$, $(u_n - u)\Phi \in \mathcal{D}_0^{1,p}(\Omega)$ and thus by Theorem 6,

$$\begin{aligned} \int_{\mathbb{R}^N} |\Phi|^p \, d\nu_n = \int_\Omega g|(u_n - u)\Phi|^p \, dx &\leq C_H \|g\| \int_\Omega |\nabla((u_n - u)\Phi)|^p \, dx \\ &= C_H \|g\| \int_{\mathbb{R}^N} |\nabla((u_n - u)\Phi)|^p \, dx. \end{aligned}$$

Take $n \to \infty$ and use Lemma 2 to obtain

$$\int_{\mathbb{R}^N} |\Phi|^p \, d\nu \leq C_H \|g\| \int_{\mathbb{R}^N} |\Phi|^p \, d\Gamma. \tag{4.1}$$

Now by Proposition 5, we get

$$\nu(E) \leq C_H \|g\| \Gamma(E) \; , \forall E \textit{ Borel in } \mathbb{R}^N. \tag{4.2}$$

In particular, $\nu \ll \Gamma$ and hence by Radon-Nikodym theorem,

$$\nu(E) = \int_E \frac{d\nu}{d\Gamma} \, d\Gamma \; , \forall E \textit{ Borel in } \mathbb{R}^N. \tag{4.3}$$

Further, by Lebesgue differentiation theorem (page 152-168 of [16]) we have

$$\frac{d\nu}{d\Gamma}(x) = \lim_{r \to 0} \frac{\nu(B_r(x))}{\Gamma(B_r(x))}. \tag{4.4}$$

Now replacing g by $g\chi_{B_r(x)}$ and proceeding as before,

$$\nu(B_r(x)) \leq C_H \left\| g\chi_{B_r(x)} \right\| \; \Gamma(B_r(x)).$$

Thus from (4.4) we get

$$\frac{d\nu}{d\Gamma}(x) \leq C_H \mathcal{C}_g(x) \tag{4.5}$$

and hence $\left\|\frac{d\nu}{d\Gamma}\right\|_\infty \leq C_H \mathcal{C}_g^*$. Now from (4.3) we obtain $\nu(E) \leq C_H\ \mathcal{C}_g^* \Gamma(E)$ for all Borel subsets E of $\mathbb{R}^N$. □

Remark 5. In [29] (for $p = 2$ and $\Omega = \mathbb{R}^N$) and in [27] (for $p \in (1, N)$ and $\Omega \subseteq \mathbb{R}^N$), the authors considered the following concentration function:

$$S_g(x) \quad = \quad \liminf_{r \to 0} \left\{ \int_\Omega |\nabla u|^p\ dx : u \in \mathcal{D}_0^{1,p}(\Omega \cap B_r(x)),\ \int_\Omega g|u|^p\ dx = 1 \right\},$$

and they considered the singular set to be $\left\{x \in \overline{\Omega} : S_g(x) < \infty\right\}$ and assumed that the closure of it, is at most countable (see (H) of [29] and (H1) of [27]). One can easily see that their singular set coincides with $\sum_g$ (by (5.1)). The countability assumption allowed them to describe ν as a countable sum of Dirac measures located on $\sum_g$ and using this they have obtained the absolute continuity of ν with respect to Γ (see Lemma 2.1 of [27] and Lemma 3.1 of [29]). Whereas we use the Radon-Nikodym theorem and the Lebesgue differentiation theorem to prove the absolute continuity of ν with respect to Γ. We would like to stress that we do not make any assumption on the cardinality or the structure of $\sum_g$ for this purpose.

The next lemma gives a lower estimate for the measure $\tilde{\Gamma}$. Similar estimate is obtained in Lemma 2.1 of [27]. We make a weaker assumption, $\overline{\sum_g}$ is of Lebesgue measure 0, than the assumption $\overline{\sum_g}$ is countable.

Lemma 4. *Let $g \in \mathcal{H}(\Omega)$ be such that $g \geq 0$ and $|\overline{\sum_g}| = 0$. If $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$, then*

$$\tilde{\Gamma} \geq \begin{cases} |\nabla u|^p + \frac{\nu}{C_H \mathcal{C}_g^*}, & \text{if } \mathcal{C}_g^* \neq 0, \\ |\nabla u|^p, & \text{otherwise.} \end{cases}$$

Proof. Our proof splits in to three steps.
Step 1: $\tilde{\Gamma} \geq |\nabla u|^p$. Let $\phi \in C_c^\infty(\mathbb{R}^N)$ with $0 \leq \phi \leq 1$, we need to show that $\int_{\mathbb{R}^N} \phi\ d\tilde{\Gamma} \geq \int_{\mathbb{R}^N} \phi |\nabla u|^p\ dx$. Notice that,

$$\int_{\mathbb{R}^N} \phi\ d\tilde{\Gamma} = \lim_{n\to\infty} \int_{\mathbb{R}^N} \phi\ d\tilde{\Gamma}_n = \lim_{n\to\infty} \int_\Omega \phi |\nabla u_n|^p\ dx = \lim_{n\to\infty} \int_\Omega F(x, \nabla u_n(x))\ dx,$$

where $F : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}$ is defined as $F(x, z) = \phi(x)|z|^p$. Clearly, F is a Caratheodory function and $F(x, .)$ is convex for almost every x. Hence, by Theorem 2.6 of [26] (page 28), we have $\displaystyle\lim_{n\to\infty} \int_\Omega \phi |\nabla u_n|^p\ dx \geq \int_\Omega \phi |\nabla u|^p\ dx = \int_{\mathbb{R}^N} \phi |\nabla u|^p\ dx$ and this proves our claim 1.

Step 2: $\tilde{\Gamma} = \Gamma$, on $\overline{\sum_g}$. Let $E \subset \overline{\sum_g}$ be a Borel set. Thus, for each $m \in \mathbb{N}$, there exists an open subset O_m containing E such that $|O_m| = |O_m \setminus E| < \frac{1}{m}$. Let $\varepsilon > 0$ be given. Then, for any $\phi \in C_c^\infty(O_m)$ with $0 \le \phi \le 1$, using (2.3) we have

$$\begin{aligned}\left|\int_\Omega \phi \, d\Gamma_n \, dx - \int_\Omega \phi \, d\tilde{\Gamma}_n \, dx\right| &= \left|\int_\Omega \phi|\nabla(u_n - u)|^p \, dx - \int_\Omega \phi|\nabla u_n|^p \, dx\right| \\ &\le \varepsilon \int_\Omega \phi|\nabla u_n|^p \, dx + \mathrm{C}(\varepsilon, p)\int_\Omega \phi|\nabla u|^p dx \\ &\le \varepsilon L + \mathrm{C}(\varepsilon, p)\int_{O_m} |\nabla u|^p \, dx,\end{aligned}$$

where $L = \sup_n \left\{\int_\Omega |\nabla u_n|^p \, dx\right\}$. Now letting $n \to \infty$, we obtain $\left|\int_\Omega \phi \, d\Gamma - \int_\Omega \phi \, d\tilde{\Gamma}\right| \le \varepsilon L + \mathrm{C}(\varepsilon, p)\int_{O_m} |\nabla u|^p \, dx$. Therefore,

$$\begin{aligned}\left|\Gamma(O_m) - \tilde{\Gamma}(O_m)\right| &= \sup\left\{\left|\int_\Omega \phi \, d\Gamma - \int_\Omega \phi \;\; d\tilde{\Gamma}\right| : \phi \in C_c^\infty(O_m), 0 \le \phi \le 1\right\} \\ &\le \varepsilon L + \mathrm{C}(\varepsilon, p)\int_{O_m} |\nabla u|^p \, dx,\end{aligned}$$

Now as $m \to \infty$, $|O_m| \to 0$ and hence $|\Gamma(E) - \tilde{\Gamma}(E)| \le \varepsilon L$. Since $\varepsilon > 0$ is arbitrary, we conclude $\Gamma(E) = \tilde{\Gamma}(E)$.

Step 3: $\tilde{\Gamma} \ge |\nabla u|^p + \frac{\nu}{C_H \mathcal{C}_g^*}$, if $\mathcal{C}_g^* \ne 0$. Let $\mathcal{C}_g^* \ne 0$. Then from Lemma 3 we have $\Gamma \ge \frac{\nu}{C_H \mathcal{C}_g^*}$. Furthermore, (4.5) and (4.3) ensures that ν is supported on $\sum_g$. Hence Step 1 and Step 2 yields the following:

$$\tilde{\Gamma} \ge \begin{cases} |\nabla u|^p, \\ \frac{\nu}{C_H \mathcal{C}_g^*}. \end{cases} \tag{4.6}$$

Since $|\overline{\sum_g}| = 0$, the measure $|\nabla u|^p$ is supported inside $\overline{\sum_g}^c$ and hence from (4.6) we easily obtain $\tilde{\Gamma} \ge |\nabla u|^p + \frac{\nu}{C_H \mathcal{C}_g^*}$. ☐

Lemma 5. *Let $g \in \mathcal{H}(\Omega)$, $g \ge 0$ and $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$ and $\Phi_R \in C_b^\infty(\mathbb{R}^N)$ with $0 \le \Phi_R \le 1$, $\Phi_R = 0$ on $\overline{B_R}$ and $\Phi_R = 1$ on B_{R+1}^c. Then,*

(A) $$\lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega\cap\overline{B_R}^c} g|u_n|^p \, dx = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \nu_n(\Omega \cap \overline{B_R}^c) = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_\Omega \Phi_R \, d\nu_n,$$

(B) $$\lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega\cap\overline{B_R}^c} |\nabla u_n|^p \, dx = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \Gamma_n(\Omega \cap \overline{B_R}^c) = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_\Omega \Phi_R \, d\Gamma_n.$$

Proof. By Brezis-Lieb lemma,

$$\overline{\lim_{n\to\infty}} \left|\nu_n(\Omega \cap \overline{B_R}^c) - \int_{\Omega\cap\overline{B_R}^c} g|u_n|^p \, dx\right| = \overline{\lim_{n\to\infty}} \left|\int_{\Omega\cap\overline{B_R}^c} g|u_n - u|^p \, dx - \int_{\Omega\cap\overline{B_R}^c} g|u_n|^p \, dx\right|$$

$$= \int_{\Omega\cap\overline{B_R}^c} g|u|^p \, dx.$$

As $g|u|^p \in L^1(\Omega)$, the right-hand side integral goes to 0 as $R \to \infty$. Thus, we get the first equality in (A). For the second equality, it is enough to observe that

$$\int_{\Omega\cap\overline{B_{R+1}}^c} g|u_n - u|^p \, dx \le \int_\Omega g|u_n - u|^p \Phi_R \, dx \le \int_{\Omega\cap\overline{B_R}^c} g|u_n - u|^p \, dx.$$

Now by taking $n, R \to \infty$ respectively we get the required equality. Now we proceed to prove (B). For $\varepsilon > 0$, there exists $\mathrm{C}(\varepsilon, p) > 0$ (by (2.3)) such that

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} \left| \Gamma_n(\Omega \cap \overline{B_R}^c) - \int_{\Omega\cap\overline{B_R}^c} |\nabla u_n|^p \, dx \right| \\
&= \overline{\lim_{n\to\infty}} \left| \int_{\Omega\cap\overline{B_R}^c} |\nabla(u_n - u)|^p \, dx - \int_{\Omega\cap\overline{B_R}^c} |\nabla u_n|^p \, dx \right| \\
&\le \varepsilon \overline{\lim_{n\to\infty}} \int_{\Omega\cap\overline{B_R}^c} |\nabla u_n|^p \, dx + \mathrm{C}(\varepsilon, p) \int_{\Omega\cap\overline{B_R}^c} |\nabla u|^p \, dx \\
&\le \varepsilon L + \mathrm{C}(\varepsilon, p) \int_{\Omega\cap\overline{B_R}^c} |\nabla u|^p \, dx \,,
\end{aligned}$$

where $L \ge \int_\Omega |\nabla u_n|^p \, dx$ for all n. Thus, by taking $R \to \infty$ and then $\varepsilon \to 0$, we obtain the first equality of (B). The second equality of part (B) follows from the same argument as that of part (A). □

Lemma 6. *Let $g \in \mathcal{H}(\Omega)$, $g \ge 0$ and $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$. Set*

$$\nu_\infty = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \nu_n(\Omega \cap \overline{B_R}^c) \quad \text{and} \quad \Gamma_\infty = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \Gamma_n(\Omega \cap \overline{B_R}^c).$$

Then

(i) $\nu_\infty \le C_H \, \mathcal{C}_g(\infty) \Gamma_\infty$,

(ii) $\overline{\lim}_{n\to\infty} \int_\Omega g|u_n|^p \, dx = \int_\Omega g|u|^p \, dx + \|\nu\| + \nu_\infty$.

(iii) Further, if $|\overline{\sum_g}| = 0$, then we have

$$\overline{\lim}_{n\to\infty} \int_\Omega |\nabla u_n|^p \, dx \ge \begin{cases} \displaystyle\int_\Omega |\nabla u|^p \, dx + \frac{\|\nu\|}{C_H \mathcal{C}_g^*} + \Gamma_\infty, & \textit{if } \mathcal{C}_g^* \ne 0 \\ \displaystyle\int_\Omega |\nabla u|^p \, dx + \Gamma_\infty, & \textit{otherwise.} \end{cases}$$

Proof. (i): For $R > 0$, choose $\Phi_R \in C_b^1(\mathbb{R}^N)$ satisfying $0 \leq \Phi_R \leq 1$, $\Phi_R = 0$ on $\overline{B_R}$ and $\Phi_R = 1$ on B_{R+1}^c. Clearly, $(u_n - u)\Phi_R \in \mathcal{D}_0^{1,p}(\Omega \cap \overline{B_R}^c)$. Since $\left\|g\chi_{\overline{B_R}^c}\right\| < \infty$, by Theorem 6,

$$\int_{\Omega \cap \overline{B_R}^c} g|(u_n - u)\Phi_R|^p \, dx \leq C_H \left\|g\chi_{\overline{B_R}^c}\right\| \int_{\Omega \cap \overline{B_R}^c} |\nabla((u_n - u)\Phi_R)|^p \, dx.$$

By Lemma 2 we have, $\overline{\lim_{n\to\infty}} \int_{\Omega \cap \overline{B_R}^c} |\nabla((u_n - u)\Phi_R)|^p \, dx = \overline{\lim_{n\to\infty}} \int_{\Omega \cap \overline{B_R}^c} |\Phi_R|^p \, d\Gamma_n$. Therefore, letting $n \to \infty$, $R \to \infty$ and using Lemma 5 successively in the above inequality we obtain $\nu_\infty \leq C_H \mathcal{C}_g(\infty) \, \Gamma_\infty$.

(ii): By choosing Φ_R as above and using Brezis-Lieb lemma together with part (A) of Lemma 5 we have,

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} \int_\Omega g|u_n|^p \, dx \\
&= \overline{\lim_{n\to\infty}} \left[\int_\Omega g|u_n|^p (1 - \Phi_R) \, dx + \int_\Omega g|u_n|^p \Phi_R \, dx \right] \\
&= \overline{\lim_{n\to\infty}} \left[\int_\Omega g|u|^p (1 - \Phi_R) \, dx + \int_\Omega g|u_n - u|^p (1 - \Phi_R) \, dx + \int_\Omega g|u_n|^p \Phi_R \, dx \right] \\
&= \int_\Omega g|u|^p \, dx + \|\nu\| + \nu_\infty.
\end{aligned}$$

(iii): Notice that

$$\begin{aligned}
\overline{\lim_{n\to\infty}} \int_\Omega |\nabla u_n|^p \, dx &= \overline{\lim_{n\to\infty}} \left[\int_\Omega |\nabla u_n|^p (1 - \Phi_R) \, dx + \int_\Omega |\nabla u_n|^p \Phi_R \, dx \right] \\
&= \tilde{\Gamma}(1 - \Phi_R) + \overline{\lim_{n\to\infty}} \int_\Omega |\nabla u_n|^p \Phi_R \, dx
\end{aligned}$$

By taking $R \to \infty$ and using part (B) Lemma 5 we get

$$\overline{\lim_{n\to\infty}} \int_\Omega |\nabla u_n|^p \, dx = \|\tilde{\Gamma}\| + \Gamma_\infty.$$

Now, using Lemma 4, we obtain

$$\overline{\lim}_{n\to\infty} \int_\Omega |\nabla u_n|^p \, dx \geq \begin{cases} \displaystyle\int_\Omega |\nabla u|^p \, dx + \frac{\|\nu\|}{C_H \mathcal{C}_g^*} + \Gamma_\infty, & \text{if } \mathcal{C}_g^* \neq 0 \\ \displaystyle\int_\Omega |\nabla u|^p \, dx + \Gamma_\infty, & \text{otherwise.} \end{cases}$$

□

Remark 6 (The assumptions on the the singular set $\sum_g$)**.** Let us recall the following fundamental result in the concentration compactness theory by Lions [21, Lemma 1.2]. Let ν, Γ be two non-negative, bounded measures on $\mathbb{R}^N$ such that

$$\left[\int_{\mathbb{R}^N} |\phi|^q \, d\nu\right]^{\frac{1}{q}} \leq C \left[\int_{\mathbb{R}^N} |\phi|^p \, d\Gamma\right]^{\frac{1}{p}}, \ \forall \, \phi \in C_c^\infty(\mathbb{R}^N), \tag{4.7}$$

for some $C > 0$ and $1 \leq p < q$. Then there exist at most a countable set $\left\{x_j \in \mathbb{R}^N : j \in \mathbb{J}\right\}$ and $\nu_j \in (0, \infty)$ such that

$$\nu = \sum_{j \in \mathbb{J}} \nu_j \delta_{x_j}. \tag{4.8}$$

For $q = p$, in [27, 29] authors assumed the countability of the singular set $\overline{\sum_g}$ and obtain the same representation of γ as in (4.8). This representation helps them for proving the results ([29, Lemma 3.1], [27, Lemma 2.1]). In this situation, we have seen that γ is supported on the set $\overline{\sum_g}$ (by Lemma 3). In this article, we relax the countability assumption on $\overline{\sum_g}$ and by pass the representation of γ in order to derive Lemma 6. Indeed, we have (4.1) which is the limiting case of (4.7) $(q = p)$.

In the following lemma we approximate $\mathcal{F}(\Omega)$ functions using $L^\infty(\Omega)$ functions similar result is obtained for $\mathcal{F}_{\frac{N}{p}}(\Omega)$ in Proposition 3.2 of [7].

Lemma 7. *$g \in \mathcal{F}(\Omega)$ if and only if for every $\epsilon > 0$, $\exists g_\epsilon \in L^\infty(\Omega)$ such that $|Supp(g_\epsilon)| < \infty$ and $\|g - g_\epsilon\| < \epsilon$.*

Proof. Let $g \in \mathcal{F}(\Omega)$ and $\epsilon > 0$ be given. By definition of $\mathcal{F}(\Omega)$, $\exists g_\epsilon \in C_c^\infty(\Omega)$ such that $\|g - g_\epsilon\| < \epsilon$. This g_ϵ fulfill our requirements. For the converse part, take a g satisfying the hypothesis. Let $\epsilon > 0$ be arbitrary. Then $\exists g_\epsilon \in L^\infty(\Omega)$ such that $|Supp(g_\epsilon)| < \infty$ and $\|g - g_\epsilon\| < \frac{\epsilon}{2}$. Thus, $g_\epsilon \in L^{\frac{N}{p}}(\Omega)$ and hence there exists $\phi_\epsilon \in C_c^\infty(\Omega)$ such that $\|g_\epsilon - \phi_\epsilon\|_{\frac{N}{p}} < \frac{\epsilon}{2C}$, where C is the embedding constant for the embedding $L^{\frac{N}{p}}(\Omega)$ into $\mathcal{H}(\Omega)$. Now by triangle inequality, we obtain $\|g - \phi_\epsilon\| < \epsilon$ as required. ☐

The next proposition gives an interesting property of capacity, which helps us to localize the norm on $\mathcal{H}(\Omega)$.

Proposition 11. *There exists $C_1, C_2 > 0$ such that for $F \subset\subset \Omega$,*

(i) $\mathrm{Cap}_p(F \cap B_r(x), \Omega \cap B_{2r}(x)) \leq C_1 \mathrm{Cap}_p(F \cap B_r(x), \Omega), \ \forall r > 0.$

(ii) $\mathrm{Cap}_p(F \cap B_{2R}^c, \Omega \cap \overline{B_R}^c) \leq C_2 \mathrm{Cap}_p(F \cap B_{2R}^c, \Omega), \ \forall R > 0.$

Proof. (*i*) Let $\Phi \in C_c^\infty(\mathbb{R}^N)$ be such that $0 \le \Phi \le 1$, $\Phi = 1$ on $\overline{B_1(0)}$ and $Supp(\Phi) \subseteq B_2(0)$. Take $\Phi_r(z) = \Phi(\frac{z-x}{r})$. Let $\epsilon > 0$ be given. Then for $F \subset\subset \Omega$, $\exists u \in \mathcal{N}(F \cap B_r(x))$ such that $\int_\Omega |\nabla u|^p < \text{Cap}_p(F \cap B_r(x), \Omega) + \epsilon$. If we set $w_r(z) = \Phi_r(z)u(z)$, then it is easy to see that $w_r \in \mathcal{D}_0^{1,p}(\Omega \cap B_{2r}(x))$ and $w_r \ge 1$ on $F \cap B_r(x)$. Further, we have the following estimate:

$$\begin{aligned}\int_\Omega |\nabla w_r|^p \, dx &\le C \left[\int_\Omega |\Phi_r|^p |\nabla u|^p \, dx + \int_\Omega |u|^p |\nabla \Phi_r|^p \, dx \right] \\ &\le C \left[\int_\Omega |\nabla u|^p \, dx + \left(\int_\Omega |u|^{p*} \, dx \right)^{p/p^*} \left(\int_\Omega |\nabla \Phi_r|^N \, dx \right)^{p/N} \right].\end{aligned}$$

By noticing $\displaystyle\int_\Omega |\nabla \Phi_r|^N \, dx \le \int_{\mathbb{R}^N} |\nabla \Phi|^N \, dx$ and then using the Sobolev embedding, we obtain

$$\int_\Omega |\nabla w_r|^p \, dx \le C_1 \int_\Omega |\nabla u|^p \, dx,$$

where C_1 is a constant independent of F, r and ϵ. Therefore,

$$\text{Cap}_p(F \cap B_r(x), \Omega \cap B_{2r}(x)) \le C_1 \text{Cap}_p(F \cap B_r(x), \Omega) + C_1 \epsilon.$$

Now as $\epsilon > 0$ is arbitrary we obtain the desired result.
(*ii*) For $\Phi \in C_b^\infty(\mathbb{R}^N)$ with $0 \le \Phi \le 1$, $\Phi = 0$ on $\overline{B_1}(0)$ and $\Phi = 1$ on $B_2(0)^c$, we take $\Phi_R(z) = \Phi(\frac{z}{R})$. The rest of the proof is similar to the proof of (*i*). □

Now we consider the map $|G| : \mathcal{D}_0^{1,p}(\Omega) \mapsto \mathbb{R}$ defined as $|G|(u) = \displaystyle\int_\Omega |g||u|^p \, dx$ and state the following proposition.

Proposition 12. *Let $g \in \mathcal{H}(\Omega)$. Then G is compact if and only if $|G|$ is compact.*

Proof. Let $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$. Then $u_n \to u$, $g|u_n|^p \to g|u|^p$ and $|g||u_n|^p \to |g||u|^p$ a.e in Ω. Further,

$$|g|u_n|^p| = |g||u_n|^p. \tag{4.9}$$

Now as $g \in \mathcal{H}(\Omega)$, both of $g|u_n|^p$, $g|u|^p$ belong to $L^1(\Omega)$. Since equality occurs in (4.9), a direct application of generalized dominated convergence theorem proves the required equivalence. □

Lemma 8. *Let $g \in \mathcal{H}(\Omega)$ and $G : \mathcal{D}_0^{1,p}(\Omega) \mapsto \mathbb{R}$ is compact. Then,*

(*i*) *if (A_n) is a sequence of bounded measurable subsets such that χ_{A_n} decreases to 0, then $\|g\chi_{A_n}\| \to 0$ as $n \to \infty$.*

(ii) $\|g\chi_{B_n^c}\| \to 0$ *as* $n \to \infty$.

Proof. (i) Let (A_n) be a sequence of bounded measurable subsets such that χ_{A_n} decreases to 0. If $\|g\chi_{A_n}\| \nrightarrow 0$, then $\exists a > 0$ such that $\|g\chi_{A_n}\| > a, \forall n$ (by the monotonicity of the norm). Thus, $\exists F_n \subset\subset \Omega$ and $u_n \in \mathcal{N}(F_n)$ such that

$$\int_\Omega |\nabla u_n|^p \, dx < \frac{1}{a} \int_{F_n \cap A_n} |g| \, dx \leq \frac{1}{a} \int_{\{|u_n| \geq 1\}} |g||u_n|^p \, dx. \tag{4.10}$$

Since A_n's are bounded and χ_{A_n} decreases to 0, it follows that $|A_n| \to 0$, as $n \to \infty$. Further, as $g \in L^1(A_1)$, we also have $\displaystyle\int_{F_n \cap A_n} |g| \, dx \to 0$. Hence from the above inequalities, $u_n \to 0$ in $\mathcal{D}_0^{1,p}(\Omega)$. For $0 < \epsilon < 1$, consider $w_n^\epsilon = \dfrac{|u_n|^p}{(|u_n| + \epsilon)^{p-1} \|u_n\|_{\mathcal{D}}}$. One can check that for each n, $w_n^\epsilon \in \mathcal{D}_0^{1,p}(\Omega)$ and it is bounded uniformly (with respect to n) in $\mathcal{D}_0^{1,p}(\Omega)$. Thus up to a sub sequence, w_n^ϵ converges weakly to w in $\mathcal{D}_0^{1,p}(\Omega)$ as $n \to \infty$. Now using the embedding of $\mathcal{D}_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$ we obtain that $\|w_n^\epsilon\|_{\frac{p^*}{p}} \leq C \dfrac{\|u_n\|_{\mathcal{D}}^{p-1}}{\epsilon^{(p-1)}}$. Thus $\|w_n^\epsilon\|_{\frac{p^*}{p}} \to 0$ as $n \to \infty$ and hence $w = 0$ i.e. $w_n^\epsilon \rightharpoonup 0$ in $\mathcal{D}_0^{1,p}(\Omega)$ as $n \to \infty$. By the compactness of $|G|$ we infer $\displaystyle\lim_{n\to\infty} \int_\Omega |g||w_n^\epsilon|^p \, dx = 0$. On the other hand, for each $n \in \mathbb{N}$ and $0 < \epsilon < 1$,

$$\begin{aligned}
\int_\Omega |g||w_n^\epsilon|^p \, dx &= \int_\Omega \frac{|g||u_n|^{p^2}}{(|u_n| + \epsilon)^{p^2-p} \|u_n\|_{\mathcal{D}}^p} \, dx \geq \int_{|u_n| \geq \epsilon} \frac{|g||u_n|^{p^2}}{(2|u_n|)^{p^2-p} \|u_n\|_{\mathcal{D}}^p} \, dx \\
&= \frac{1}{2^{p^2-p}} \int_{\{|u_n| \geq \epsilon\}} \frac{|g||u_n|^p}{\|u_n\|_{\mathcal{D}}^p} \, dx > \frac{a}{2^{p^2-p}}
\end{aligned}$$

which is a contradiction.
(ii) If $\|g\chi_{B_n^c}\| \nrightarrow 0$, as $n \to \infty$, then there exists $F_n \subset\subset \Omega$ such that

$$a < \frac{\int_{F_n \cap B_n^c} |g| \, dx}{\mathrm{Cap}_p(F_n, \Omega)} \leq \frac{\int_{F_n \cap B_n^c} |g| \, dx}{\mathrm{Cap}_p(F_n \cap B_n^c, \Omega)} \leq \frac{C \int_{F_n \cap B_n^c} |g| \, dx}{\mathrm{Cap}_p(F_n \cap B_n^c, \Omega \cap \overline{B}_{\frac{n}{2}}^c)}$$

for some $a > 0$ and $C > 0$. Last inequality follows from the part (ii) of Proposition 11. Thus, for each n there exists $z_n \in \mathcal{D}_0^{1,p}(\Omega \cap \overline{B}_{\frac{n}{2}}^c)$ with $z_n \geq 1$ on $F_n \cap B_n^c$ such that

$$\int_\Omega |\nabla z_n|^p \, dx < \frac{C}{a} \int_{F_n \cap B_n^c} |g| \, dx \leq \frac{C}{a} \int_\Omega |g||z_n|^p \, dx.$$

By taking $w_n = \dfrac{z_n}{\|z_n\|_{\mathcal{D}}}$ and following a same argument as in (i) we contradict the compactness of $|G|$ and hence, that of G. ∎

Next for $\phi \in C_c^\infty(\Omega)$ we compute $\mathcal{C}_\phi$.

Proposition 13. *Let $\phi \in C_c^\infty(\Omega)$. Then $\mathcal{C}_\phi \equiv 0$.*

Proof. First notice that for $\phi \in C_c^\infty(\Omega)$,

$$\left\|\phi \chi_{B_r(x)}\right\| = \sup_{F \subset\subset \Omega} \left[\frac{\int_{F\cap B_r(x)} |\phi| \; dx}{\mathrm{Cap}_p(F, \Omega)} \right] \leq \sup_{F\subset\subset\Omega} \left[\frac{\sup(|\phi|) |(F \cap B_r)^\star|}{\mathrm{Cap}_p((F\cap B_r)^\star)} \right].$$

If d is the radius of $(F \cap B_r)^\star$ then

$$\frac{|(F\cap B_r)^\star|}{\mathrm{Cap}_p((F\cap B_r)^\star)} = \frac{\omega_N d^N}{N\omega_N (\frac{N-p}{p-1})^{p-1} d^{(N-p)}} = C(N,p) d^p \leq C(N,p) r^p.$$

Thus, $\mathcal{C}_\phi(x) = \lim_{r\to 0}\left\|\phi\chi_{B_r(x)}\right\| = 0$. Also, one can easily see that $\mathcal{C}_\phi(\infty) = 0$ as ϕ has compact support . ∎

Remark 7. In fact, the same arguments as in the above proposition shows that $\mathcal{C}_g \equiv 0$ if $g \in L^\infty(\Omega)$ and g has compact support.

The next theorem proves Theorem 1, Theorem 2 and Theorem 3 in one shot.

Theorem 8. *Let $g \in \mathcal{H}(\Omega)$. Then the following statements are equivalent:*

(i) $G : \mathcal{D}_0^{1,p}(\Omega) \mapsto \mathbb{R}$ *is compact,*

(ii) g *has absolute continuous norm in* $\mathcal{H}(\Omega)$*,*

(iii) $g \in \mathcal{F}(\Omega)$*,*

(iv) $\mathcal{C}_g^* = 0 = \mathcal{C}_g(\infty)$*.*

Proof. $(i) \implies (ii)$: Let G be compact. Take a sequence of measurable subsets (A_n) of Ω such that χ_{A_n} decreases to 0 a.e. in Ω. Part (ii) of Lemma 8 gives $\left\|g\chi_{B_n^c}\right\| \to 0$, as $n \to \infty$. Choose $\epsilon > 0$ arbitrarily. There exists $N_0 \in \mathbb{N}$, such that $\left\|g\chi_{B_n^c}\right\| \leq \frac{\epsilon}{2}, \forall n \geq N_0$. Now $A_n = (A_n \cap B_{N_0}) \cup (A_n \cap B_{N_0}^c)$, for each n. Thus,

$$\|g\chi_{A_n}\| \leq \left\|g\chi_{A_n\cap B_{N_0}}\right\| + \left\|g\chi_{A_n\cap B_{N_0}^c}\right\| \leq \left\|g\chi_{A_n\cap B_{N_0}}\right\| + \frac{\epsilon}{2}.$$

By part (i) of Lemma 8, there exists $N_1 (\geq N_0) \in \mathbb{N}$ such that $\left\|g\chi_{A_n\cap B_{N_0}}\right\| \leq \frac{\epsilon}{2}$, $\forall n \geq N_1$ and hence $\|g\chi_{A_n}\| \leq \epsilon$ for all $n \geq N_1$. Therefore, g has absolutely continuous norm.

$(ii) \implies (iii)$: Let g has absolute continuous norm in $\mathcal{H}(\Omega)$. Then, $\left\|g\chi_{B_m^c}\right\|$ converge to 0 as $m \to \infty$. Let $\epsilon > 0$ be arbitrary. We choose $m_\varepsilon \in \mathbb{N}$ such that $\left\|g\chi_{B_m^c}\right\| < \epsilon$, $\forall m \geq m_\varepsilon$. Now for any $n \in \mathbb{N}$,

$$g = g\chi_{\{|g|\leq n\}\cap B_{m_\varepsilon}} + g\chi_{\{|g|>n\}\cap B_{m_\varepsilon}} + g\chi_{B_{m_\varepsilon}^c} := g_n + h_n.$$

where $g_n = g\chi_{\{|g|\leq n\}\cap B_{m_\varepsilon}}$ and $h_n = g\chi_{\{|g|>n\}\cap B_{m_\varepsilon}} + g\chi_{B_{m_\varepsilon}^c}$. Clearly, $g_n \in L^\infty(\Omega)$ and $|Supp(g_n)| < \infty$. Furthermore,

$$\|h_n\| \leq \left\|g\chi_{\{|g|>n\}\cap B_{m_\varepsilon}}\right\| + \left\|g\chi_{B_{m_\varepsilon}^c}\right\| < \left\|g\chi_{\{|g|>n\}\cap B_{m_\varepsilon}}\right\| + \epsilon\,.$$

Now, $g \in L^1_{loc}(\Omega)$ ensures that $\chi_{\{|g|>n\}\cap B_{m_\varepsilon}} \to 0$ as $n \to \infty$. As g has absolutely continuous norm, $\left\|g\chi_{\{|g|>n\}\cap B_{m_\varepsilon}}\right\| < \epsilon$ for large n. Therefore, $\|h_n\| < 2\epsilon$ for large n. Hence, Lemma 7 concludes that $g \in \mathcal{F}(\Omega)$.

$(iii) \implies (iv)$: Let $g \in \mathcal{F}(\Omega)$ and $\epsilon > 0$ be arbitrary. Then there exists $g_\varepsilon \in C_c^\infty(\Omega)$ such that $\|g - g_\varepsilon\| < \epsilon$. Thus Proposition 13 infers that $\mathcal{C}_{g_\varepsilon}$ vanishes. Now as $g = g_\varepsilon + (g - g_\varepsilon)$, it follows that $\mathcal{C}_g(x) \leq \mathcal{C}_{g_\varepsilon}(x) + \mathcal{C}_{g-g_\varepsilon}(x) \leq \|g - g_\varepsilon\| < \epsilon$ and hence $\mathcal{C}_g^* = 0$. By a similar argument one can show $\mathcal{C}_g(\infty) = 0$.

$(iv) \implies (i)$: Assume that $\mathcal{C}_g^* = 0 = \mathcal{C}_g(\infty)$. Let (u_n) be a bounded sequence in $\mathcal{D}_0^{1,p}(\Omega)$. Then by Lemma 6, up to a sub-sequence we have,

$$\begin{aligned} \nu_\infty &\leq C_H\, \mathcal{C}_g(\infty)\Gamma_\infty, \\ \|\nu\| &\leq C_H \mathcal{C}_g^* \|\Gamma\|, \\ \lim_{n\to\infty} \int_\Omega |g||u_n|^p\, dx &= \int_\Omega |g||u|^p\, dx + \|\nu\| + \nu_\infty. \end{aligned}$$

As $\mathcal{C}_g^* = 0 = \mathcal{C}_g(\infty)$ we immediately conclude that $\lim_{n\to\infty} \int_\Omega |g||u_n|^p\, dx = \int_\Omega |g||u|^p\, dx$ and hence $G : \mathcal{D}_0^{1,p}(\Omega) \mapsto \mathbb{R}$ is compact (Proposition 12). □

Remark 8 (Rellich compactness theorem)**.** Let Ω be a bounded domain in $\mathbb{R}^N$ and $g \equiv 1$ on Ω. Then, by Remark 7, $\mathcal{C}_g \equiv 0$ and hence, by the above equivalence G is compact on $\mathcal{D}_0^{1,p}(\Omega)$ i.e., $\mathcal{D}_0^{1,p}(\Omega)$ is compactly embedded into $L^p(\Omega)$.

Remark 9. Let $N > p$ and $g(x) = \frac{1}{|x|^p}$ in $\mathbb{R}^N$. Then for any $r > 0$, using Proposition 4 we get

$$\frac{\int_{B_r(0)} \frac{dx}{|x|^p}}{\mathrm{Cap}_p(B_r(0))} = \frac{(p-1)^{p-1}}{(N-p)^p}.$$

Thus $\mathcal{C}_g(0) = \frac{(p-1)^{p-1}}{(N-p)^p}$ and hence $g \notin \mathcal{F}(\mathbb{R}^N)$.

Remark 10. Let $X = (X(\Omega), \|.\|_X)$ be a Banach function space and $f \in X$. Then f is said to have continuous norm in X, if for each $x \in \Omega$, $\left\|f\chi_{B_r(x)}\right\|$ converges to 0, as $r \to 0$. Observe that by Theorem 8, the set of all functions having continuous norm and the set of all function having absolute continuous norm are one and the same on $\mathcal{H}(\Omega)$. However, in [18], authors constructed a Banach function space where these two sets are different.

Now, we recall Maz'ya's concentration function Π_g ,(see Section 2.4.2, page 130 of [25]). For $F \subset\subset \Omega$ with $|F| \neq 0$, let $\Pi(F, g, \Omega) := \frac{\int_F |g| \, dx}{\mathrm{Cap}_p(F,\Omega)}$. Then

$$\Pi_g(x) = \lim_{r \to 0} \sup\{\Pi(g, F, \Omega) : F \subset\subset \Omega \cap B_r(x)\},$$
$$\Pi_g(\infty) = \lim_{r \to \infty} \sup\left\{\Pi(g, F, \Omega) : F \subset\subset \Omega \cap B_r(0)^c\right\}.$$

Next proposition shows that $\mathcal{C}_g$ coincides with Π_g. As $\mathcal{C}_g$ measures the concentration using the norm of $\mathcal{H}(\Omega)$, we prefer $\mathcal{C}_g$ over Π_g.

Proposition 14. *Let $g \in \mathcal{H}(\Omega)$. Then $\mathcal{C}_g(x) = \Pi_g(x)$ for all $x \in \overline{\Omega}$ and $\mathcal{C}_g(\infty) = \Pi_g(\infty)$.*

Proof. First notice that $\Pi_g(x) \leq \mathcal{C}_g(x)$, for any $x \in \overline{\Omega}$ and $\Pi_g(\infty) \leq \mathcal{C}_g(\infty)$. On other hand for $V \subset\subset \Omega$,

$$\frac{\int_V |g|\chi_{B_r(x)} \, dx}{\mathrm{Cap}_p(V,\Omega)} \leq \frac{\int_{V \cap B_r(x)} |g| \, dx}{\mathrm{Cap}_p(V \cap B_r(x), \Omega)} \leq \sup_{F \subset\subset \Omega \cap B_{2r}(x)} \left[\frac{\int_F |g| \, dx}{\mathrm{Cap}_p(F,\Omega)}\right] = \Pi_g^{2r}(x).$$

The last inequality follows as $V \cap B_r(x)$ is relatively compact in $\Omega \cap B_{2r}(x)$. Taking the supremum over all $V \subset\subset \Omega$ and letting $r \to 0$ we obtain $\mathcal{C}_g(x) \leq \Pi_g(x)$. By a similar argument we also get $\mathcal{C}_g(\infty) \leq \Pi_g(\infty)$ as required. ∎

5. A concentration compactness criteria

Recall that, for $g \in \mathcal{H}(\Omega)$, the best constant B_g in (1.1) is given by

$$\frac{1}{B_g} = \inf_{u \in G^{-1}\{1\}} \int_\Omega |\nabla u|^p \, dx.$$

Proof of Theorem 4. Let $(u_n) \in G^{-1}\{1\}$ be a sequence that minimizes $\int_\Omega |\nabla u|^p \, dx$ over $G^{-1}\{1\}$. Then up to a sub-sequence we can assume that $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$ and $u_n \to u$ a.e. in Ω. Further, $|\nabla u_n - \nabla u|^p \overset{*}{\rightharpoonup} \Gamma$, $|\nabla u_n| \overset{*}{\rightharpoonup} \tilde{\Gamma}$, $g|u_n - u|^p \overset{*}{\rightharpoonup} \nu$ in $\mathbb{M}(\mathbb{R}^N)$. Since $u_n \in G^{-1}\{1\}$, using Lemma 6 we have

$$1 = \int_\Omega g|u|^p \, dx + \|\nu\| + \nu_\infty.$$

Suppose $\|\nu\|$ or ν_∞ is nonzero. Then $\mathcal{C}_g^*$ or $\mathcal{C}_g(\infty) \neq 0$ respectively. Now using Hardy-Sobolev inequality and Lemma 6, we obtain the following estimate:

$$\begin{aligned}
1 = B_g \times \overline{\lim}_{n\to\infty} \int_\Omega |\nabla u_n|^p \, dx &\geq B_g \left[\int_\Omega |\nabla u|^p \, dx + \frac{\|\nu\|}{C_H \mathcal{C}_g^*} + \Gamma_\infty \right] \\
&\geq B_g \left[\frac{1}{B_g} \int_\Omega g|u|^p \, dx + \frac{\|\nu\|}{C_H \mathcal{C}_g^*} + \frac{\nu_\infty}{C_H \mathcal{C}_g(\infty)} \right] \\
&> \frac{B_g}{B_g} \left[\int_\Omega g|u|^p \, dx + \|\nu\| + \nu_\infty \right],
\end{aligned}$$

a contradiction. Thus $\|\nu\| = 0 = \nu_\infty$. Therefore, $\displaystyle\int_\Omega g|u|^p \, dx = 1$ and consequently, B_g is attained at u.

Remark 11. For $g(x) = \frac{1}{|x|^p}$ in $\mathbb{R}^N$, it is well known that B_g is not attained in $\mathcal{D}_0^{1,p}(\Omega)$. Further, $\mathcal{C}_g(0) = \frac{(p-1)^{p-1}}{(N-p)^p}$ and hence $C_H \mathcal{C}_g^* = B_g$.

Remark 12. Recall the definition of $S_g(x)$. In [27], author also considered the following quantities :

$$\begin{aligned}
S_g(x) &:= \lim_{r\to 0} \inf \left\{ \int_\Omega |\nabla u|^p \, dx : u \in \mathcal{D}_0^{1,p}(\Omega \cap B_r(x)), \ \int_\Omega g|u|^p \, dx = 1 \right\}, \\
S_g^* &:= \sup_{x\in\overline{\Omega}} S_g(x), \\
S_g(\infty) &:= \lim_{R\to\infty} \inf \left\{ \int_\Omega |\nabla u|^p \, dx : u \in \mathcal{D}_0^{1,p}(\Omega \cap B_R^c), \ \int_\Omega g|u|^p \, dx = 1 \right\}, \\
S_g &:= \inf \left\{ \int_\Omega |\nabla u|^p \, dx : u \in \mathcal{D}_0^{1,p}(\Omega), \ \int_\Omega g|u|^p \, dx = 1 \right\}.
\end{aligned}$$

Since $S_g(.)$ captures the best constant in the Hardy inequality locally at the points of Ω and at the infinity, by (2.2), we have

$$\|g\| \leq \frac{1}{S_g} \leq C_H \|g\|, \quad \mathcal{C}_g^* \leq \frac{1}{S_g^*} \leq C_H \mathcal{C}_g^*, \quad \mathcal{C}_g(\infty) \leq \frac{1}{S_g(\infty)} \leq C_H \mathcal{C}_g(\infty). \tag{5.1}$$

Therefore, if $C_H \mathcal{C}_g^* < \|g\|$ and $C_H \mathcal{C}_g(\infty) < \|g\|$ then $S_g < S_g^*$ and $S_g < S_g(\infty)$. Thus, if in addition $\overline{\sum_g}$ is countable, then Theorem 4 also follow from Theorem 3.1 of [27]. Therefore, our sufficient condition is slightly weaker than that of [27]. This is mainly because of the gap in the Hardy inequality given in 6 (see (2.2)). However, on the other hand, our sufficient condition assumes $|\overline{\sum_g}| = 0$ instead of its countability.

Remark 13. In [27], Smets proved the Mazya's compactness criteria by showing that G is compact if and only if $S_g^* = S_g(\infty) = \infty$. Observe that, one can easily derive this result by using (5.1) together with Theorem 3.

Proof of Theorem 5. Let $h \in \mathcal{H}(\Omega)$ be non-negative and $|\overline{\sum_h}| = 0$. Take a non-zero, non-negative $\phi \in \mathcal{F}(\Omega)$ and $\epsilon_0 = \frac{(2C_H-1)\|h\|}{\|\phi\|}$, then for $\epsilon > \epsilon_0$, let $g = h + \epsilon\phi$. Clearly, $|\overline{\sum_g}| = 0$ and

$$C_H \mathcal{C}_g^* = C_H \mathcal{C}_{h+\epsilon\phi}^* = C_H \mathcal{C}_h^* \le C_H \|h\| < \frac{\|h\| + \epsilon\|\phi\|}{2} \le \|g\| \le B_g.$$

Similarly, we can show $C_H \mathcal{C}_g(\infty) < \|g\| \le B_g$. Therefore, by Theorem 4, B_g is attained.

Remark 14. (i). For $2 \le k < N$ and for $z \in \mathbb{R}^N$, we write $z = (x,y) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. Now consider $g(z) = \frac{1}{|x|^p}$ in $\mathbb{R}^k \times \mathbb{R}^{N-k}$. By Theorem 2.1 of [8], $g \in \mathcal{H}(\mathbb{R}^N)$ if $p < k$. Next we show that $\sum_g = \{0\} \times \mathbb{R}^{N-k}$. For any $(0,y) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ and $r > 0$, using the translation invariance of both the integral and the Cap_p, we have

$$\frac{\int_{B_r(0,y)} g(z)\, dz}{\mathrm{Cap}_p(B_r(0,y))} = \frac{\int_{B_r(0,0)} \frac{1}{|x|^p}\, dz}{\mathrm{Cap}_p(B_r(0,0))} \ge \frac{\int_{B_r(0,0)} \frac{1}{|z|^p}\, dz}{\mathrm{Cap}_p(B_r(0,0))}.$$

Now by taking $r \to 0$ we have $\mathcal{C}_g(0,y) \ge \mathcal{C}_{\frac{1}{|z|^p}}((0,0)) > 0$ and hence $\sum_g \supseteq \{0\} \times \mathbb{R}^{N-k}$. Next for $z_0 = (x_0, y_0) \notin \{0\} \times \mathbb{R}^{N-k}$, let $0 < r < |x_0.|$ Then by Proposition 4 we obtain

$$\frac{\int_{B_r(z_0)} \frac{1}{|x|^p}\, dz}{\mathrm{Cap}_p(B_r(z_0))} \le \frac{\frac{1}{(|x_0|-r)^p} \int_{B_r(z_0)} dz}{\mathrm{Cap}_p(B_r(z_0))} = \left(\frac{p-1}{N-p}\right)^{p-1} \left(\frac{r^p}{N(|x_0|-r)^p}\right).$$

Now by taking $r \to 0$, we obtain $\mathcal{C}_g(z_0) = 0$. Hence, $\sum_g = \{0\} \times \mathbb{R}^{N-k}$.
(ii). Let $2 \le k < N$, $p < k$. We consider $g(z) = \frac{1}{|x|^p}$, for $z = (x,y) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. In Example 14, we have seen that $g \in \mathcal{H}(\Omega)$ with $\sum_g$ is uncountable and $|\overline{\sum_g}| = 0$. Now choose any $\phi \in \mathcal{F}(\Omega)$ such that $\|\phi\| = 2(2C_H - 1)\|g\|$ and consider $\tilde{g} := g + \phi$. Then $\epsilon_0 = \frac{1}{2}$ and hence that $B_{\tilde{g}}$ is attained (by Theorem 5). Further, in Example 14, we have seen that $\sum_{\tilde{g}}$ is also uncountable and $|\overline{\sum_{\tilde{g}}}| = 0$. Thus, $\tilde{g}$ lies outside the class of functions considered in [27, 29].

Corollary 2. *Let $g \in \mathcal{H}(\Omega)$ with $g \ge 0$ and $|\overline{\sum_g}| = 0$. If $C_H dist(g, \mathcal{F}(\Omega)) < \|g\|$, then B_g is attained in $\mathcal{D}_0^{1,p}(\Omega)$.*

Proof. For $g, h \in L^1_{loc}(\Omega)$ and $F \subset\subset \Omega$,

$$\frac{\int_F |g| \chi_{B_r(x)}\, dx}{\mathrm{Cap}_p(F,\Omega)} \le \frac{\int_F |g-h| \chi_{B_r(x)}\, dx}{\mathrm{Cap}_p(F,\Omega)} + \frac{\int_F |h| \chi_{B_r(x)}\, dx}{\mathrm{Cap}_p(F,\Omega)}.$$

By taking the supremum over all such F and r tends to 0 respectively, we obtain $\mathcal{C}_g(x) \le \mathcal{C}_{g-h}(x) + \mathcal{C}_h(x)$ and hence

$$\mathcal{C}_g^* \le \mathcal{C}_{g-h}^* + \mathcal{C}_h^*. \tag{5.2}$$

Now as $C_H dist(g, \mathcal{F}(\Omega)) < \|g\|$, $\exists \phi \in \mathcal{F}(\Omega)$ such that $C_H\|g-\phi\| < \|g\|$. Thus by (5.2), $C_H\mathcal{C}_g^* \leq C_H\mathcal{C}_{g-\phi}^* \leq C_H\|g-\phi\| < \|g\| \leq B_g$ and similarly $C_H\mathcal{C}_g(\infty) < B_g$. Now the result follows from Theorem 4. ☐

Next proposition also gives us another way to produce the Hardy potential for which B_g is attained in $\mathcal{D}_0^{1,p}(\Omega)$ without G being compact.

Proposition 15. *Let $g \in L^1_{loc}(\Omega)$ be such that $g^+ \in \mathcal{F}(\Omega)$. Then the best constant B_g is attained.*

Proof. Let (u_n) be a sequence that minimizes $\displaystyle\int_\Omega |\nabla u|^p \, dx$ over $G^{-1}\{1\}$. Then (u_n) is bounded in $\mathcal{D}_0^{1,p}(\Omega)$ and hence up to a subsequence $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$ and $u_n \to u$ a.e. in Ω. Since $g^+ \in \mathcal{F}(\Omega)$, $\displaystyle\lim_{n\to\infty}\int_\Omega g^+|u_n|^p \, dx = \int_\Omega g^+|u|^p \, dx$ (by Theorem 1). Further, $\displaystyle\int_\Omega g^-|u_n|^p \, dx = \int_\Omega g^+|u_n|^p \, dx - 1$. Now Fatous lemma gives $\displaystyle\int_\Omega g^-|u|^p \, dx \leq \int_\Omega g^+|u|^p \, dx - 1$. Thus $1 \leq \int_\Omega g|u|^p \, dx$ and hence $\tilde{u} := \frac{u}{[\int_\Omega g|u|^p \, dx]^{1/p}}$ is a required element in $\mathcal{D}_0^{1,p}(\Omega)$ for which the best constant is attained. ☐

Remark 15. The above proposition gives an alternate way to produce examples of weight function g for which the best constant B_g is attained without G being compact. For example, take g in $L^1_{loc}(\Omega)$ with $g^+ \in \mathcal{F}(\Omega)$ and $g^- \notin \mathcal{F}(\Omega)$.

6. Acknowledgement

T. V. Anoop would like to thank the Department of Science & Technology, India for the research grant DST/INSPIRE/04/2014/001865.

References

[1] Adimurthi, N. Chaudhuri, and M. Ramaswamy. An improved Hardy-Sobolev inequality and its application. *Proc. Amer. Math. Soc.*, 130(2):489–505 (electronic), 2002.

[2] W. Allegretto. Principal eigenvalues for indefinite-weight elliptic problems in $\mathbb{R}^n$. *Proc. Amer. Math. Soc.*, 116(3):701–706, 1992.

[3] A. Alvino. Sulla diseguaglianza di Sobolev in spazi di Lorentz. *Boll. Un. Mat. Ital. A (5)*, 14(1):148–156, 1977.

[4] T. V. Anoop. Weighted eigenvalue problems for the p-Laplacian with weights in weak Lebesgue spaces. *Electron. J. Differential Equations*, pages No. 64, 22, 2011.

[5] T. V. Anoop, U. Das, and A. Sarkar. On the generalized Hardy-Rellich inequalities. *Proc. Roy. Soc. Edinburgh Sect. A*, 150(2):897–919, 2020.

[6] T. V. Anoop, P. Drábek, and S. Sasi. Weighted quasilinear eigenvalue problems in exterior domains. *Calc. Var. Partial Differential Equations*, 53(3-4):961–975, 2015.

[7] T. V. Anoop, M. Lucia, and M. Ramaswamy. Eigenvalue problems with weights in Lorentz spaces. *Calc. Var. Partial Differential Equations*, 36(3):355–376, 2009.

[8] M. Badiale and G. Tarantello. A Sobolev-Hardy inequality with applications to a nonlinear elliptic equation arising in astrophysics. *Arch. Ration. Mech. Anal.*, 163(4):259–293, 2002.

[9] C. Bennett and R. Sharpley. *Interpolation of operators*, volume 129 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.

[10] G. Bertin. *Dynamics of galaxies.* Cambridge University Press, Cambridge, 2000.

[11] H. Brézis and E. a. Lieb. A relation between pointwise convergence of functions and convergence of functionals. *Proc. Amer. Math. Soc.*, 88(3):486–490, 1983.

[12] H. Brezis and J. L. Vázquez. Blow-up solutions of some nonlinear elliptic problems. *Rev. Mat. Univ. Complut. Madrid*, 10(2):443–469, 1997.

[13] L. Ciotti. Dynamical models in astrophysics. *Lecture Notes, Scuola Normale Superiore, Pisa*, 2001.

[14] D. E. Edmunds and W. D. Evans. *Hardy operators, function spaces and embeddings.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2004.

[15] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions.* Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.

[16] H. Federer. *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[17] S. Filippas and A. Tertikas. Optimizing improved Hardy inequalities. *J. Funct. Anal.*, 192(1):186–233, 2002.

[18] J. Lang and A. Nekvinda. A difference between continuous and absolutely continuous norms in Banach function spaces. *Czechoslovak Math. J.*, 47(122)(2):221–232, 1997.

[19] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.

[20] P. L. Lions. The concentration-compactness principle in the calculus of variations: The locally compact cases I & II. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1(2):109–145, 223–283, 1984.

[21] P. L. Lions. The concentration-compactness principle in the calculus of variations: The limit cases I & II. *Rev. Mat. Iberoamericana*, 1(1):45–121, 145–201, 1985.

[22] G. G. Lorentz. Some new functional spaces. *Ann. of Math. (2)*, 51:37–55, 1950.

[23] G. Mancini, I. Fabbri, and K. Sandeep. Classification of solutions of a critical Hardy-Sobolev operator. *J. Differential Equations*, 224(2):258–276, 2006.

[24] A. Manes and A. M. Micheletti. Un'estensione della teoria variazionale classica degli autovalori per operatori ellittici del secondo ordine. *Boll. Un. Mat. Ital. (4)*, 7:285–301, 1973.

[25] V. G. Maz'ja. *Sobolev spaces.* Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985. Translated from the Russian by T. O. Shaposhnikova.

[26] F. Rindler. *Calculus of variations.* Universitext. Springer, Cham, 2018.

[27] D. Smets. A concentration-compactness lemma with applications to singular eigenvalue problems. *J. Funct. Anal.*, 167(2):463–480, 1999.

[28] A. Szulkin and M. Willem. Eigenvalue problems with indefinite weight. *Studia Math.*, 135(2):191–201, 1999.

[29] A. Tertikas. Critical phenomena in linear elliptic problems. *J. Funct. Anal.*, 154(1):42–66, 1998.

[30] N. Visciglia. A note about the generalized Hardy-Sobolev inequality with potential in $L^{p,d}(\mathbb{R}^n)$. *Calc. Var. Partial Differential Equations*, 24(2):167–184, 2005.

[31] A. C. Zaanen. *An introduction to the theory of integration.* North-Holland Publishing Company, Amsterdam, 1958.

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