REGULARITY OF POWERS OF BIPARTITE GRAPHS

A. V. JAYANTHAN, N. NARAYANAN, AND S. SELVARAJA

Abstract. Let G be a finite simple graph and $I(G)$ denote the corresponding edge ideal. For all $s \geq 1$, we obtain upper bounds for $\operatorname{reg}(I(G)^s)$ for bipartite graphs. We then compare the properties of G and G', where G' is the graph associated with the polarization of the ideal $(I(G)^{s+1} : e_1 \cdots e_s)$, where $e_1, \ldots e_s$ are edges of G. Using these results, we explicitly compute $\operatorname{reg}(I(G)^s)$ for several subclasses of bipartite graphs.

1. Introduction

Let $G = (V(G), E(G))$ denote a finite simple undirected graph with vertices $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$. By identifying the vertices with the variables in the polynomial ring $k[x_1, \ldots, x_n]$, where k is a field, we can associate to each graph G a monomial ideal $I(G)$ generated by the set $\{x_i x_j \mid \{x_i, x_j\} \in E(G)\}$. The ideal $I(G)$ is called the *edge ideal* of G. This notion was introduced by Villarreal in [29]. Since then, the researchers have been investigating the connection between the combinatorial properties of the graphs and the algebraic properties of the corresponding edge ideals. In particular, there have been active research on bounding the homological invariants of edge ideals in terms of the combinatorial invariants of the associated graphs, see for example [3], [10], [12], [15], [17], [19], [22], [23], [26], [28], [32], [33]. In this article, we study the Castelnuovo-Mumford regularity of powers of edge ideals of bipartite graphs. For a homogeneous ideal I, we denote by $\operatorname{reg}(I)$, the Castelnuovo-Mumford regularity, henceforth called regularity, of I.

It was proved by Cutkosky, Herzog and Trung, [9], and independently by Kodiyalam [20], that for a homogeneous ideal I in a polynomial ring, $\operatorname{reg}(I^s)$ is a linear function for $s \gg 0$, i.e., there exist integers a, b, s_0 such that

$$\operatorname{reg}(I^s) = as + b \text{ for all } s \geq s_0.$$

It is known that a is bounded above by the maximum of degree of elements in a minimal generating set of I. But a general bound for b as well as s_0 is unknown. In this paper, we consider $I = I(G)$, the edge ideal of G. In this case, there exist integers b and s_0 such that $\operatorname{reg}(I^s) = 2s + b$ for all $s \geq s_0$. Our objective in this paper is to find b and s_0 in terms of combinatorial invariants of the graph G. We refer the reader to [5] for a review of results in the literature which identify classes of edge ideals for which b and s_0 are explicitly computed.

It is known that for any graph G,

$$\nu(G) + 1 \leq \operatorname{reg}(I(G)) \leq \text{co-chord}(G) + 1, \tag{1.1}$$

where $\nu(G)$ denote the induced matching number of G and co-chord(G) denote the co-chordal cover number of G. The lower bound was proved by Katzman, [18] and the upper bound was proved by Woodroofe, [32]. Beyarslan, Hà and Trung proved that for any graph G and $s \geq 1$, $2s + \nu(G) - 1 \leq \operatorname{reg}(I(G))^s$, [5]. They also proved that the equality holds for edge ideals of forests (for all $s \geq 1$) and cycles (for all $s \geq 2$). Moghimian, Sayed and Yassemi have shown that the equality holds for edge ideals of whiskered cycles as well, [25].

There is no general upper bound known for $\operatorname{reg}(I(G)^s)$. Woodroofe's inequality, (1.1), suggests $\operatorname{reg}(I(G)^s) \leq 2s + \text{co-chord}(G) - 1$ for all $s \geq 1$. We prove this inequality for bipartite graphs. Using this we discover new classes of graphs for which b and s_0 can be computed explicitly. We determine several classes of graphs for which the equality

$$\operatorname{reg}(I(G)^s) = 2s + b$$

holds, for every $s \geq 1$, i.e., $s_0 = 1$ and b is explicitly described using combinatorial invariants associated with G. One of the central ideas in our proofs is the comparison of certain properties of a graph G with those of another associated graph G'. Let G be a graph and $e_1, \ldots, e_s$ be edges (not necessarily distinct) of G, $s \geq 1$. Banerjee, in [3], introduced the notion of *even-connection* with respect to the s-fold product $e_1 \cdots e_s$, see Definition 2.2. He showed that $(I(G)^{s+1} : e_1 \cdots e_s)$ is a quadratic monomial ideal, and hence its polarization corresponds to a graph, say G'. Then Banerjee showed that G' is the union of G with all the even-connections with respect to the s-fold product $e_1 \cdots e_s$. Though G' has been obtained from G through an algebraic operation, some of the combinatorial properties seem to be comparable. The ideal $I(G')$ has emerged as a good tool in the study of asymptotic regularity of the edge ideals, see [1], [3], [5], [23]. The comparison between G and G' (equivalently, between $I(G)$ and $I(G')$) provides a tool to compute upper bound for the regularity. If G is an arbitrary graph, e is an edge in G and G' is the graph associated with the polarization of $(I(G)^2 : e)$, then one of the main results states that $\text{co-chord}(G') \leq \text{co-chord}(G)$ (Theorem 3.2). Alilooee and Banerjee proved that if G is bipartite, then so is G', [1]. Also, Banerjee proved that $\operatorname{reg}(I(G)^{s+1}) \leq \max\{\operatorname{reg}(I(G') + 2s, \operatorname{reg}(I(G)^s\}$, [3]. We use these results to get an upper bound for the regularity of $I(G)^s$ when G is a bipartite graph:

Theorem 1.1 (Theorem 3.6). *Let G be a bipartite graph. Then for all $s \geq 1$, we have*

$$\operatorname{reg}(I(G)^s) \leq 2s + \text{co-chord}(G) - 1. \tag{1.2}$$

We also compare certain properties and invariants, algebraic as well as combinatorial, of G and G' for several subclasses of bipartite graphs. We prove that if G is either unmixed bipartite (Theorem 4.1) or P_k-free bipartite (Theorem 4.3) or nK_2-free (Corollary 4.7), then so is G'. We also prove that the induced matching number of G' is at most that of G (Proposition 4.4). As a consequence, we obtain an upper bound for $\operatorname{reg}(I(G)^s)$ when G is a bipartite graph (Corollary 4.5). Comparison between the graphs G and G' yields yet another positive result, namely, a partial answer to a question posed by Banerjee, [2, Question 6.2.2], on classifying all graphs G and edges $e_1 \cdots e_s$ such that

$$\operatorname{reg}(I(G)) \geq \operatorname{reg}(I(G)^{s+1} : e_1 \cdots e_s) \text{ for all } s \geq 1.$$

We obtain some sufficient conditions for this inequality to be true, (Proposition 4.10) and as a consequence we prove that this inequality holds true for unmixed bipartite graph, chordal bipartite, whiskered bipartite graph, bipartite P_6-free graphs and connected bipartite graphs with regularity equal to three.

We then move on to compute precise expressions for the regularity of powers of edge ideals. In [5], the authors raised the question, for which graphs G, $\operatorname{reg}(I(G)^s) = 2s + \nu(G) - 1$ for $s \gg 0$. We observe that for certain classes of bipartite graphs, the induced matching number coincides with the co-chordal cover number, for example, unmixed bipartite, chordal bipartite and whiskered bipartite. We then use the upper bound (1.2) for such classes of graphs to get $\operatorname{reg}(I(G)^s) = 2s + \nu(G) - 1$ for all $s \geq 1$, (Corollary 5.1). As an immediate consequence, we derive one of the main results of [5], that the above equality holds for forests. We also derive the main result of Alilooee and Banerjee, in [1], that equality holds true for connected bipartite graphs G with $\operatorname{reg}(I(G)) = 3$.

The classes of graphs discussed earlier have the property $\nu(G) = \text{co-chord}(G)$. For bipartite P_6-free graphs, it not known whether this equality holds true. However we prove that for such graphs, for all $s \geq 1$, $\operatorname{reg}(I(G)^s) = 2s + \nu(G) - 1$. It was shown by Jacques, [17], that if $n \equiv 2(mod\ 3)$, then $\operatorname{reg}(I(C_n)) = \nu(C_n) + 2$. And, Beyarslan, Hà and Trung proved that $\operatorname{reg}(I(C_n)^s) = 2s + \nu(C_n) - 1$ for all $s \geq 2$, [5]. If G is the disjoint union of $C_{n_1}, \ldots, C_{n_m}$ and k edges, for some $k \geq 1$, then we obtain a precise expression for $\operatorname{reg}(I(G)^s)$, (Theorem 5.5). We also construct, for each $t \geq 1$, a graph G_t such that $\operatorname{reg}(I(G_t)^s) - [2s + \nu(G_t) - 1] = t$, (Example 5.7).

2. Preliminaries

In this section, we set up the basic definitions and notation needed for the main results. Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. A subgraph $H \subseteq G$ is called *induced* if $\{u, v\}$ is an edge of H if and only if u and v are vertices of H and $\{u, v\}$ is an edge of G. For a vertex u in a graph G, let $N_G(u) = \{v \in V(G) | \{u, v\} \in E(G)\}$ be the set of neighbors of u. The *complement* of a graph G, denoted by G^c, is the graph on the same vertex set in which $\{u, v\}$ is an edge of G^c if and only if it is not an edge of G. A subset X of $V(G)$ is called *independent* if there is no edge $\{x, y\} \in E(G)$ for $x, y \in X$. The independence number $\alpha(G)$ is the maximum size of an independent set. Let C_k denote the cycle on k vertices and P_k denote the path on k vertices. The *length* of a path, or cycle is its number of edges.

A graph G is called *bipartite* if there are two disjoint independent subsets X, Y of $V(G)$ such that $X \cup Y = V(G)$.

Let G be a graph. We say n non-adjacent edges $\{f_1, \ldots, f_n\}$ form an nK_2 in G if G does not have an edge with one endpoint in f_i and the other in f_j for all $i, j \in \{1, \ldots, n\}$ and $i \neq j$. A graph without nK_2 is called nK_2-free. If n is 2, then $2K_2$-free graph also called *gap-free* graph. It is easy to see that, G is gap-free if and only if G^c contains no induced C_4. Thus, G is gap-free if and only if it does not contain two vertex-disjoint edges as an induced subgraph.

A *matching* in a graph G is a subgraph consisting of pairwise disjoint edges. The largest size of a matching in G is called its matching number and denoted by $\operatorname{c}(G)$ and the minimum matching number of G, denoted by $\operatorname{b}(G)$, is the minimum cardinality of the maximal matchings of G. If the subgraph is an induced subgraph, the matching is an *induced matching*. The largest size of an induced matching in G is called its induced matching number and denoted by $\nu(G)$.

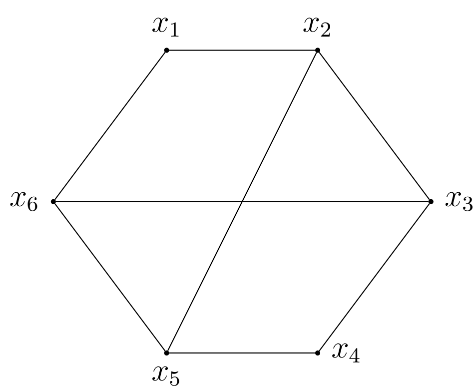
Let G be a graph. A subset $C \subseteq V(G)$ is a vertex cover of G if for each $e \in E(G)$, $e \cap C \neq \phi$. If C is minimal with respect to inclusion, then C is called *minimal vertex cover* of G. A graph G is called *unmixed (also called well-covered)* if all minimal vertex covers of G have the same number of elements.

For a graph G on n vertices, let $W(G)$ be the *whiskered graph* on $2n$ vertices obtained by adding a pendent vertex (an edge to a new vertex of degree 1) to every vertex of G.

A graph G is *weakly chordal* if every induced cycle in both G and G^c has length at most 4 and G is *chordal bipartite* if it is simultaneously weakly chordal and bipartite. Equivalently, a bipartite graph is chordal bipartite if and only if it has no induced cycle on six or more vertices.

For any undefined terminology and further basic properties of graphs, we refer the reader to [31].

Example 2.1. *Let G be the graph with vertices $V(G) = \{x_1, \ldots, x_6\}$ given below.*



Then $\Big\{\{x_2, x_3\}, \{x_5x_6\}\Big\}$ forms a matching, but not an induced matching since the induced subgraph with vertices $\{x_2, x_3, x_5, x_6\}$ contains edges $\{x_3, x_6\}$ and $\{x_2, x_5\}$. The set $\{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}\}$ forms a matching of G and $\{\{x_2, x_5\}, \{x_3, x_6\}\}$ also form a matching, the set $\{x_1, x_3, x_5\}$ forms an independent set of G. It is not hard to verify that $\mathrm{c}(G) = 3, \mathrm{b}(G) = 2, \nu(G) = 1$ and $\alpha(G) = 3$. It can also be noted that $\{x_2, x_3, x_5, x_6\}$ and $\{x_2, x_4, x_6\}$ are minimal vertex covers of G. Therefore G is not unmixed.

We recall the definition of even-connectedness and some of its important properties from [3].

Definition 2.2. *Let G be a graph. Two vertices u and v (u may be same as v) are said to be even-connected with respect to an s-fold products $e_1 \cdots e_s$, where e_i's are edges of G, not necessarily distinct, if there is a path $p_0p_1 \cdots p_{2k+1}$, $k \geq 1$ in G such that:*

1. $p_0 = u, p_{2k+1} = v$.
2. *For all* $0 \leq \ell \leq k-1$, $p_{2\ell+1}p_{2\ell+2} = e_i$ *for some* i.
3. *For all* i, $|\ \{\ell \geq 0 \mid p_{2\ell+1}p_{2\ell+2} = e_i\}\ | \leq |\ \{j \mid e_j = e_i\}\ |$.
4. *For all* $0 \leq r \leq 2k$, p_rp_{r+1} *is an edge in* G.

Example 2.3. *Let $I(G) = (x_1x_2, x_1x_5, x_2x_5, x_2x_3, x_3x_4, x_4x_5) \subset k[x_1, \ldots, x_5]$. Then $(I(G)^2 : x_2x_5) = I(G) + (x_1^2, x_1x_3, x_1x_4)$. Note that, x_1 is even-connected to itself and $\{x_1, x_3\}$, $\{x_1, x_4\}$ are even-connected with respect to x_2x_5.*

The following theorem, due to Banerjee, is used repeatedly throughout this paper.

Theorem 2.4. [3, Theorem 5.2] *For any finite simple graph G and any $s \geq 1$, let the set of minimal monomial generators of $I(G)^s$ be $\{m_1, \ldots, m_k\}$, then*

$$\operatorname{reg}(I(G)^{s+1}) \leq \max\{\operatorname{reg}(I(G)^{s+1} : m_\ell) + 2s,\ 1 \leq \ell \leq k,\ \operatorname{reg}(I(G)^s)\}.$$

Next theorem describes all the minimal generating set of an ideal $(I(G)^{s+1} : M)$, where M is minimal generator of $I(G)^s$ for $s \geq 1$.

Theorem 2.5. [3, Theorem 6.1 and Theorem 6.7] *Let G be a graph with edge ideal $I = I(G)$, and let $s \geq 1$ be an integer. Let M be a minimal generator of I^s. Then $(I^{s+1} : M)$ is minimally generated by monomials of degree 2, and uv (u and v may be the same) is a* *minimal generator of* $(I^{s+1} : M)$ *if and only if either* $\{u, v\} \in E(G)$ *or* u *and* v *are even-connected with respect to* M*.*

Further, Alilooee and Banerjee studied the even-connection in the context of bipartite graphs and showed that they behave well under even-connections.

Theorem 2.6. [1, Proposition 3.5] *Let* G *be a bipartite graph and* $s \geq 1$ *be an integer. Then for every* s*-fold product* $e_1 \cdots e_s$*,* $(I(G)^{s+1} : e_1 \cdots e_s)$ *is a quadratic squarefree monomial ideal. Moreover the graph* G' *associated to* $(I(G)^{s+1} : e_1 \cdots e_s)$ *is bipartite on the same vertex set and same bipartition as* G*.*

Polarization is a process to obtain a squarefree monomial ideal from a given monomial ideal. For details of polarization we refer to [21], [24].

Definition 2.7. *Let* $f = x_1^{m_1} \cdots x_n^{m_n}$ *be a monomial in* $R = k[x_1, \ldots, x_n]$*. Let* $\widetilde{R} = k[x_{11}, x_{12}, \ldots, x_{21}, x_{22}, \ldots, x_{n1}, x_{n_2}, \ldots]$*. Then a polarization of* f *in* $\widetilde{R}$ *is the squarefree monomial* $\widetilde{f} = x_{11} \cdots x_{1m_1} x_{21} \cdots x_{2m_2} \cdots x_{n1} \cdots x_{nm_n}$*. If* $f_1, \cdots, f_m \in R$ *are monomials and* $I = (f_1, \cdots, f_m)$*, then we call the squarefree monomial ideal* $\widetilde{I}$ *generated by the polarization of the* f_i*'s in a larger polynomial ring* $\widetilde{R}$*, the polarization of* I*.*

Let G be a graph and $I(G)$ denote the edge ideal of G. Then for any $s \geq 1$ and edges $e_1, \ldots, e_s$ of G, $\widetilde{I} = \widetilde{(I(G)^{s+1} : e_1 \cdots e_s)}$ is a squarefree quadratic monomial ideal, by Theorem 2.5. Hence there exists a graph G' associated to $\widetilde{I}$. Note also that G is a subgraph of G'.

Example 2.8. *Let* $G = C_3$ *and* $I(G) = (x_1x_2, x_2x_3, x_1x_3) \subset k[x_1, x_2, x_3]$*. Then* $I = (I(G)^2 : x_1x_3) = I(G) + x_2^2$*. Therefore,* $\widetilde{I} \subset k[x_1, x_2, x_3, x_4]$ *is given by* $\widetilde{I} = I(G) + (x_2x_4)$*. Then* G' *is given the graph* G *with the edge* $\{x_2, x_4\}$ *attached to* G*.*

Let M be a graded $R = k[x_1, \ldots, x_n]$ module. For non-negative integers i, j, let $\beta_{ij}(M)$ denote the (i, j)-th graded Betti number of M.

Theorem 2.9. [21, Proposition 1.3.4] [24, Exercise 3.15] *Let* $I \subseteq R = k[x_1, \ldots, x_n]$ *be a monomial ideal. If* $\widetilde{I} \subseteq \widetilde{R}$ *is a polarization of* I*, then for all* ℓ, j*,* $\beta_{\ell,j}(R/I) = \beta_{\ell,j}(\widetilde{R}/\widetilde{I})$*. In particular* $\operatorname{reg}(R/I) = \operatorname{reg}(\widetilde{R}/\widetilde{I})$*.*

3. Upper bound for the regularity of powers of bipartite graph ideals

In this section, we study the powers of the edge ideals of bipartite graphs. We obtain an upper bound for the regularity of powers of edge ideals of bipartite graphs in terms of co-chordal cover number.

Definition 3.1. *A graph* G *is chordal (also called triangulated) if every induced cycle in* G *has length* 3*, and is co-chordal if the complement graph* G^c *is chordal.*

The co-chordal cover number, denoted co-chord(G)*, is the minimum number* n *such that there exist co-chordal subgraphs* $H_1, \ldots, H_n$ *of* G *with* $E(G) = \bigcup_{i=1}^n E(H_i)$*.*

In the following, we relate the co-chordal cover number of a graph with that of its polarization. As a consequence we obtain an upper bound for $\operatorname{reg}(I(G)^s)$ for $s \geq 1$, when G is a bipartite graph.

Theorem 3.2. *Let G be a graph and e be an edge of G. Let G' be the graph associated to $(\widetilde{I(G)^2 : e})$. Then*

$$\text{co-chord}(G') \leq \text{co-chord}(G).$$

Proof. Let co-chord$(G) = n$. Then there exist co-chordal subgraphs $H_1, \ldots, H_n$ such that $E(G) = \bigcup_{m=1}^{n} E(H_m)$. If $G = G'$, then we are done. Let $\{p_1, p_2\} = e$, $N_G(p_1) \setminus \{p_2\} = \{p_{1,1}, \ldots, p_{1,s}\}$ and $N_G(p_2) \setminus \{p_1\} = \{p_{2,1}, \ldots, p_{2,t}\}$. For any two vertices x, y, set

$$\{[x, y]\} = \begin{cases} \{x, y\} & \text{if } x \neq y; \\ \{x, z_x\} & \text{if } x = y, \text{ where } z_x \text{ is a new vertex.} \end{cases}$$

Note that for $x, y \in V(G)$, x is even-connected to y with respect to e in G if and only if $\{x, y\} = \{p_{1,i}, p_{2,j}\}$ for some $1 \leq i \leq s, 1 \leq j \leq t$. Therefore

$$E(G') = E(G) \cup \{[p_{1,1}, p_{2,1}]\} \cup \cdots \cup \{[p_{1,1}, p_{2,t}]\} \cup \cdots \cup \{[p_{1,s}, p_{2,1}]\} \cdots \{[p_{1,s}, p_{2,t}]\}.$$

For each $1 \leq \mu \leq s$, $\{p_1, p_{1,\mu}\} \in E(H_m)$ for some $1 \leq m \leq n$. We add certain even-connected edges to H_m with a rule as described below, to get a new graph H'_m:

Since H_m is co-chordal, by [4, Lemma 1 and Theorem 2], there is an ordering of edges of H_m, $f_1 < \cdots < f_{t_m}$, such that for $1 \leq r \leq t_m$, $(V(H_m), \{f_1, \ldots, f_r\})$ has no induced subgraph isomorphic to $2K_2$.

If for $1 \leq \mu \leq s$, $\{p_{1,\mu}, p_1\} = f_k$ for some $1 \leq k \leq t_m$, then set

$$\cdots < f_k < \{[p_{1,\mu}, p_{2,1}]\} < \cdots < \{[p_{1,\mu}, p_{2,t}]\} < f_{k+1} < \cdots.$$

Then we have $E(G') = \bigcup_{m=1}^{n} E(H'_m)$. We claim that H'_m is co-chordal. Let $E(H'_m) = \{g_1, \ldots, g_{t_{m_1}}\}$ be edge set of H'_m and linearly ordered as given above. By Lemma 1 and Theorem 2 of [4], it is enough to prove that for $1 \leq r' \leq t_{m_1}$, $(V(H'_m), \{g_1, \ldots, g_{r'}\})$ has no induced subgraph isomorphic to $2K_2$. Suppose H'_m is not co-chordal. Then there exists a least i such that $(V(H'_m), \{g_1, \ldots, g_i\})$ has an induced $2K_2$-subgraph, say $\{g_j, g_i\}$. Since H_m is co-chordal, g_j and g_i cannot be in $E(H_m)$ simultaneously.

Case 1: Suppose $g_j \in E(H'_m) \setminus E(H_m)$ and $g_i = \{x_\alpha, x_\beta\} \in E(H_m)$. Let $g_j = \{[p_{1,k}, p_{2,\ell}]\}$, for some $1 \leq k \leq s$ and $1 \leq \ell \leq t$. By construction, we have

$$g_{j'} = \{p_{1,k}, p_1\} < g_j < g_i.$$

Since $g_{j'}, g_i \in E(H_m)$, they cannot form an induced $2K_2$-subgraph of H_m. Therefore, either $g_{j'}$ and g_i have a vertex in common or there exist an edge $g_h \in E(H_m)$ such that $g_h < g_i$ connecting $g_{j'}$ and g_i. If $g_{j'}$ and g_i have a vertex in common, then this contradicts the assumption that $\{g_j, g_i\}$ form an induced $2K_2$-subgraph. Suppose g_h is a an edge connecting $g_{j'}$ and g_i. Let $g_h = \{p_1, x_\alpha\}$ with $x_\alpha \neq p_2$. Then $x_\alpha \in N_{H_m}(p_1)$ and hence by construction, there is a new edge $\{[x_\alpha, p_{2,\ell}]\} \in E(H'_m)$ with the ordering

$$g_h < \{[x_\alpha, p_{2,\ell}]\} < g_i.$$

This also contradicts the assumption that $\{g_j, g_i\}$ is an induced $2K_2$-subgraph. Now if $g_h = \{p_1, x_\alpha\}$ and $x_\alpha = p_2$, then $x_\beta \in N_{H_m}(p_2)$. Therefore there is an edge $\{[p_{1,k}, x_\beta]\} \in E(H'_m)$ with the ordering

$$g_{j'} < \{[p_{1,k}, x_\beta]\} < g_i.$$

This also contradicts the assumption that $\{g_j, g_i\}$ is an induced $2K_2$-subgraph. Similarly, if $g_h = \{p_1, x_\beta\}, \{p_{1,k}, x_\alpha\}$ or $\{p_{1,k}, x_\beta\}$ for some k, then one arrives at a contradiction.

If $g_j \in E(H_m)$ and $g_i \in E(H'_m) \setminus E(H_m)$, then we get contradiction in a similar manner.

Case 2: Suppose $g_j, g_i \in E(H'_m) \setminus E(H_m)$. Let $g_i = \{[p_{1,k'}, p_{2,\ell'}]\}$ and $g_j = \{[p_{1,k}, p_{2,\ell}]\}$, for some $1 \leq k, k' \leq s$, $1 \leq \ell, \ell' \leq t$. By construction, we have

$$g_{j'} = \{p_{1,k}, p_1\} < g_j < g_{i'} = \{p_{1,k'}, p_1\} < g_i.$$

Since $p_{1,k} \in N_G(p_1)$ and $p_{2,\ell'} \in N_G(p_2)$, by construction, there exists the edge $\{[p_{1,k}, p_{2,\ell'}]\}$ in H'_m with the ordering

$$g_{j'} < \{[p_{1,k}, p_{2,\ell'}]\} < g_{i'}.$$

This contradicts the assumption that $\{g_j, g_i\}$ is an induced $2K_2$-subgraph.

Therefore H'_m is a co-chordal graph for $1 \leq m \leq n$ and $E(G') = E(H'_1) \cup \cdots \cup E(H'_n)$. Hence co-chord$(G') \leq n$. $\square$

In [1, Corollary 3.6], Alilooee and Banerjee proved that, if the minimal free resolution of $I(G)$ is linear, then so is the minimal free resolution of $(I(G)^{s+1} : e_1 \cdots e_s)$. By applying this result recursively, one can see that if the minimal free resolution of $I(G)$ is linear, then so is the minimal free resolution of $(I^2 : e_{i_1})^2 : \cdots)^2 : e_{i_j})$. We reprove this result as a consequence of Theorem 3.2.

Corollary 3.3. *Let G be any graph with edge ideal $I = I(G)$ and $e_1, \ldots, e_s$, $s \geq 1$ be some edges of G which are not necessarily distinct. If the minimal free resolution of I is linear, then so is the minimal free resolution of $(((I^2 : e_{i_1})^2 : e_{i_2})^2 \cdots)^2 : e_{i_m})$, where $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, s\}$.*

Proof. Fröberg proved that $I(G)$ has a linear minimal free resolution if and only if co-chord$(G) = 1$, [12, Theorem 1]. Let G'_1 be the graph associated with the polarization of $(I(G)^2 : e_{i_1})$. Therefore from Theorem 3.2, co-chord$(G'_1) = 1$. For $j \geq 2$, define G'_j to be the graph associated with the polarization of $(I(G'_{j-1})^2 : e_{i_j})$. Now recursively applying Theorem 3.2 and [12, Theorem 1], we get the assertion. $\square$

The following corollary helps to obtain upper bound for the asymptotic regularity of edge ideals of bipartite graphs.

Corollary 3.4. *Let G be a bipartite graph and $e_1, \ldots, e_s$, $s \geq 1$, be some edges of G which are not necessarily distinct. Let G' be the graph associated to $(I(G)^{s+1} : e_1 \cdots e_s)$. Then*

$$\text{co-chord}(G') \leq \text{co-chord}(G).$$

Proof. Since G is a bipartite graph, it follows by Theorem 2.6 that the graph G' associated to $\overbrace{(((I(G)^2 : e_1)}^{2} : \cdots)^2 : e_s)$ is bipartite on the same vertex set and same bipartition as on G. By [1, Lemma 3.7], $(((I(G)^2 : e_1)^2 : \cdots)^2 : e_s) = (I(G)^{s+1} : e_1 \cdots e_s)$. Therefore by applying Theorem 3.2 recursively, we get co-chord$(G') \leq$ co-chord(G). $\square$

If G is not a bipartite graph, then the equality $(((I(G)^2 : e_1)^2 : \cdots)^2 : e_s) = (I(G)^{s+1} : e_1 \cdots e_s)$ need not necessarily hold, see the example below. This example further shows that $(((I(G)^2 : e_1))^2 : e_2)$ has a linear minimal free resolution need not necessarily imply that $(I(G)^3 : e_1 e_2)$ has a linear minimal free resolution.

Example 3.5. *Let* $I = (x_1x_7, x_1x_2, x_2x_3, x_2x_6, x_3x_4, x_3x_5, x_4x_5, x_6x_8) \subset R = k[x_1, \ldots, x_8]$ *and* G *be the associated graph. Let* G_1 *and* G_2 *be the graphs associated to*

$$(I^3 : \widetilde{x_2x_3x_4x_5}) = I + (x_1x_3, x_1x_5, x_1x_4, x_3y_1, x_3x_6, x_4x_6, x_5x_6) \subset R_1 = R[y_1]$$

and

$$(((\widetilde{I^2 : \widetilde{x_2x_3}}))^2 : x_4x_5) = (I^3 : \widetilde{x_2x_3x_4x_5}) + (x_6y_3, x_1x_6, x_1y_2) \subset R_1[y_2, y_3]$$

respectively. Then it can easily be seen that G_1^c *is not chordal and* G_2^c *is chordal. Therefore by* [12, Theorem 1], $I(G_1)$ *does not have linear minimal free resolution and* $I(G_2)$ *has linear minimal free resolution. By Theorem 2.9,* $(((I^2 : x_2x_3))^2 : x_4x_5)$ *has linear minimal free resolution but* $(I^3 : x_2x_3x_4x_5)$ *does not have linear minimal free resolution.*

We now prove an upper bound for $\operatorname{reg}(I(G)^s)$, when G is a bipartite graph.

Theorem 3.6. *Let* G *be a bipartite graph. Then for all* $s \geq 1$,

$$\operatorname{reg}(I(G)^s) \leq 2s + \text{co-chord}(G) - 1.$$

Proof. We prove by induction on s. If $s = 1$, then the assertion follows from [32, Theorem 1]. Assume $s > 1$. By applying Theorem 2.4 and using induction, it is enough to prove that for edges $e_1, \ldots, e_s$ of G (not necessarily distinct), $\operatorname{reg}(I(G)^{s+1} : e_1 \cdots e_s) \leq \text{co-chord}(G) + 1$ for all $s > 1$. Let G' be the graph associated to the ideal $(I(G)^{s+1} : e_1 \cdots e_s)$.

$$\begin{aligned} \operatorname{reg}((I(G)^{s+1} : e_1 \cdots e_s)) &\leq \text{co-chord}(G') + 1, \\ &\leq \text{co-chord}(G) + 1, \end{aligned}$$

where the first inequality follows from [32, Theorem 1] and the second inequality follows from Corollary 3.4. Hence $\operatorname{reg}(I(G)^s) \leq 2s + \text{co-chord}(G) - 1$ for all $s \geq 1$. $\square$

Remark 3.7. *The inequality given in Theorem 3.6 could be asymptotically strict. For example, if* $G = C_8$*, then one can see that the co-chordal subgraphs of* C_8 *are paths with at most* 3 *edges so that* $\text{co-chord}(G) = 3$*. On the other hand, by* [5, Theorem 5.2], $\operatorname{reg}(I(G)^s) = 2s + 1 < 2s + 2$ *for all* $s \geq 2$.

From [5, Theorem 4.5] and Theorem 3.6 for any bipartite graph G, we have

$$2s + \nu(G) - 1 \leq \operatorname{reg}(I(G)^s) \leq 2s + \text{co-chord}(G) - 1 \text{ for any } s \geq 1. \tag{3.1}$$

As a consequence of (3.1), we derive the following result of Alilooee and Banerjee:

Corollary 3.8. [1, Proposition 2.15] *Let* G *be a bipartite graph. Then following are equivalent*

(1) $I(G)$ *has a linear presentation.*
(2) $I(G)^s$ *has a linear resolution for all* $s \geq 1$.
(3) G^c *is chordal.*

Proof. It is known that for a graph G, if G is bipartite, then $\nu(G) = 1$ if and only if $\text{co-chord}(G) = 1$. Therefore the equivalence of the three statements follow directly from (3.1) and [26, Proposition 1.3]. $\square$

For any graph G, Hà and Van Tuyl proved that $\operatorname{reg}(I(G)) \leq \mathrm{c}(G)+1$, [15, Theorem 6.7]. Woodroofe then proved a strong generalization of their result, namely, $\operatorname{reg}(I(G)) \leq \mathrm{b}(G)+1$ for any graph G, [32, Theorem 2]. Let G be a graph and $\{z_1, \ldots, z_t\}$ be a minimum maximal matching of G. Let Z_i be the subgraph of G with $E(Z_i) = z_i \cup \{$ adjacent edges of $z_i\}$. Then for each i, Z_i is a co-chordal subgraph of G and $E(G) = \cup_{i=1}^t E(Z_i)$. Hence co-chord$(G) \leq \mathrm{b}(G)$. Therefore, for any bipartite graph G, it follows from [5, Theorem 4.6] and Theorem 3.6 that

$$2s + \nu(G) - 1 \leq \operatorname{reg}(I(G)^s) \leq 2s + \mathrm{b}(G) - 1. \tag{3.2}$$

A *dominating induced matching* of G is an induced matching which also forms a maximal matching of G. If G has a dominating induced matching, then $\nu(G) = \mathrm{b}(G)$. Hence for any bipartite graph G with dominating induced matching, we have,

$$\operatorname{reg}(I(G)^s) = 2s + \nu(G) - 1 \text{ for } s \geq 1.$$

In [16], Hibi et al. characterized graphs with dominating induced matchings and also G satisfying $\nu(G) = \mathrm{b}(G)$.

4. Relation between G and G'

Let G be a graph and $e_1, \ldots, e_s$ be edges of G. Let G' be the graph associated with $(I(G)^{s+1} : e_1 \cdots e_s)$. In this section, we compare certain algebraic and combinatorial properties of G and G'. Using these comparisons we obtain upper bounds for the regularity of powers of edge ideals of bipartite graphs.

We begin by considering unmixed bipartite graphs.

Theorem 4.1. *If G is an unmixed bipartite graph, then so is the graph G' associated to $(I(G)^{s+1} : e_1 \cdots e_s)$, for any s-fold product $e_1 \cdots e_s$ and $s \geq 1$.*[1]

Proof. We prove the result using induction on s. Let G be an unmixed bipartite graph. Then by [30, Theorem 1.1], there exists a partition $V_1 = \{x_1, \ldots, x_n\}$ and $V_2 = \{y_1, \ldots, y_n\}$ with $V(G) = V_1 \cup V_2$. First we show that the graph G' associated to $(I(G)^2 : e)$ is an unmixed bipartite graph for any edge e in G. By Theorem 2.6, $(I(G)^2 : e)$ is bipartite on the same vertex set having the same bipartition as in G. Since $\{x_i, y_i\} \in E(G')$ for all i, by [30, Theorem 1.1] we need to show that $\{x_i, y_k\} \in E(G')$, if $\{x_i, y_j\}, \{x_j, y_k\} \in E(G')$ for distinct i, j, k.

Case I: Suppose $\{x_i, y_j\}, \{x_j, y_k\} \in E(G)$. Since G is an unmixed bipartite graph, there is an edge $\{x_i, y_k\} \in E(G)$, by [30, Theorem 1.1]. Hence $\{x_i, y_k\} \in E(G')$.

Case II: Suppose $\{x_i, y_j\} \in E(G)$ and $\{x_j, y_k\} \notin E(G)$. Let $x_j p_1 p_2 y_k$ be an even-connection between x_j and y_k with respect to $e = p_1 p_2$. Since $\{x_i, y_j\}, \{x_j, p_1\} \in E(G)$, by [30, Theorem 1.1], there is an edge $\{x_i, p_1\} \in E(G)$. Therefore there is an even-connection $x_i p_1 p_2 y_k$ with respect to e. Hence $\{x_i, y_k\} \in E(G')$.

Case III: If $\{x_i, y_j\} \notin E(G)$ and $\{x_j, y_k\} \in E(G)$. Let $x_i p_1 p_2 y_j$ be an even-connection between x_i and y_j with respect to $e = p_1 p_2$. Since $\{p_2, y_j\}, \{x_j, y_k\} \in E(G)$, by [30, Theorem 1.1], there is an edge $\{p_2, y_k\} \in E(G)$. Therefore there is an even-connection $x_i p_1 p_2 y_k$, with respect to e. Hence $\{x_i, y_k\} \in E(G')$.

[1]In a personal communication, we have been informed that Banerjee and Mukundan have also obtained Theorem 4.1.

Case IV: If $\{x_i, y_j\}, \{x_j, y_k\} \notin E(G)$. Consider the even-connections $x_i p_1 p_2 y_j$ and $x_j p_1 p_2 y_k$ between x_i, y_j and x_j, y_k respectively with respect to e. Then there is an even-connection $x_i p_1 p_2 y_k$ between x_i and y_k with respect to e. Therefore $\{x_i, y_k\} \in E(G')$. Hence G' is an unmixed bipartite graph.

Assume by induction that for any unmixed bipartite graph G, $(I(G)^s : e_1 \cdots e_{s-1})$ is an unmixed bipartite graph for any $(s-1)$-fold product and $s > 1$. By [1, Lemma 3.7], we have $(I(G)^{s+1} : e_1 \cdots e_s) = \big((I(G)^2 : e_i)^s : \prod_{j \neq i} e_j\big)$. By the case $s = 1$, the graph associated to $(I(G)^2 : e_i)$ is an unmixed bipartite graph, say G_i. Therefore by induction the graph associated to $(I(G_i)^s : \prod_{j \neq i} e_j)$ is an unmixed bipartite graph. This completes the proof of the theorem.

□

The following example shows that Theorem 4.1 is not true if the graph is not bipartite.

Example 4.2. *Let $I = (x_1x_4, x_1x_2, x_1x_3, x_2x_3, x_2x_5, x_3x_6) \subset R = k[x_1, \ldots, x_6]$ and G be the graph associated to I. Then G is unmixed, but not bipartite. Taking $e_1 = \{x_1, x_2\}$, we get $\widetilde{(I^2 : e_1)} = I + (x_4x_5, x_3x_5, x_3x_4, x_3y_1) \subset R[y_1]$. Let G' be the graph associated with $\widetilde{(I^2 : e_1)}$. Then it can be seen that, (x_1, x_3, x_5) and $(x_1, x_2, x_4, x_5, x_6, y_1)$ are minimal vertex covers of G'. Therefore, G' is not unmixed.*

A graph G is called H-free for some graph H if G does not contain an induced subgraph isomorphic to H. Bıyıkoğlu and Civan proved that if G is bipartite P_6-free, then $\operatorname{reg}(I(G)) = \nu(G) + 1$, [6, Theorem 3.15]. Below we prove that if G is P_6-free, then so is G'. This result is crucial in obtaining a precise expression, in the next section, for the regularity of $I(G)^s$.

Theorem 4.3. *If G is a bipartite P_k-free graph for some $k \geq 4$, then so is the graph G' associated to $(I(G)^{s+1} : e_1 \cdots e_s)$, for any s-fold product $e_1 \cdots e_s$ and $s \geq 1$.*

Proof. By [1, Lemma 3.7], it is enough to prove the assertion for $s = 1$. Let G be a bipartite P_k-free graph, for some $k \geq 4$. First we show that the graph G' associated to $(I(G)^2 : e)$ is P_k-free for any edge e in G. Note that G' is bipartite on the same vertex set. Suppose G' has an induced path $P_k : x_1x_2 \cdots x_k$. First assume that $E(P_k) \setminus E(G)$ has only one edge, say $\{x_i, x_{i+1}\}$. Let the even-connection be $x_i p_1 p_2 x_{i+1}$, where $e = \{p_1, p_2\}$. Note that $\{x_i, x_{i+1}\} \cap \{p_1, p_2\} = \emptyset$. We first show that the vertices p_1 and p_2 cannot be equal to or adjacent to x_j for $j < i-1$ and $j > i+2$.

Claim I: $p_1 \neq x_j$ for $j \neq i-1$ and $p_2 \neq x_j$ for $j \neq i+2$.
If $p_1 = x_j$ for some $j \neq i-1$, then $\{x_j, x_i\} \in E(G)$. This contradicts the assumption that P_k is an induced path. Similarly if $p_2 = x_j$ for some $j \neq i+2$, then $\{x_j, x_{i+1}\} \in E(G)$, which is again a contradiction.

Claim II: $\{p_1, x_j\} \notin E(G)$ for any $j \neq i, i+2$ and $\{p_2, x_j\} \notin E(G)$ for any $j \neq i-1, i+1$. Suppose $\{p_1, x_j\} \in E(G)$ for some $j \neq i, i+2$. Then $x_j p_1 p_2 x_{i+1}$ is an even-connection between x_j and x_{i+1} so that $\{x_j, x_{i+1}\} \in E(G')$. Since $j \neq i, i+2$, this is a contradiction to the assumption that P_k is an induced path. Similarly, it can be seen that $\{p_2, x_j\} \notin E(G)$ for $j \neq i-1, i+1$.

Now we deal with the remaining possibilities. If $p_1 = x_{i-1}$, then $p_2 \neq x_{i+2}$ and hence we have a path $P' : x_{i-1} p_2 x_{i+1} x_{i+2}$ in G. Similarly, if $p_2 = x_{i+2}$, then $p_1 \neq x_{i-1}$ and hence we get a path $P' : x_{i-1}x_ip_1x_{i+2}$. Now suppose $\{p_1, x_{i+2}\} \in E(G)$. If $\{p_2, x_{i-1}\} \in E(G)$, then there is an even-connection $x_{i-1}p_2p_1x_{i+2}$ which is a contradiction. Therefore $\{p_2, x_{i-1}\} \notin E(G)$. Hence we have a path $P' : x_{i-1}x_ip_1x_{i+2}$ in G. If $\{p_2, x_{i-1}\} \in E(G)$, then $\{p_1, x_{i+2}\} \notin E(G)$ and hence we have a path $P' : x_{i-1}p_2x_{i+1}x_{i+2}$ in G.

Since p_1 and p_2 cannot be equal to x_j for $j < i-1$ and $j > i+2$, replace the segment $x_{i-1}x_ix_{i+1}x_{i+2}$ in P_k with P', to obtain an induced path $x_1 \cdots x_{i-2}P'x_{i+3} \cdots x_k$ of length $k-1$ in G which contradicts our hypothesis that G is P_k-free.

Now suppose $E(P_k) \setminus E(G)$ has more than one edge. By the proof of Claim I and Claim II, there cannot be more than two pairs of vertices which are even-connected. Moreover, if there are two even-connections, then the evenly connected vertices have to be $\{x_{i-1}, x_i\}$ and $\{x_i, x_{i+1}\}$ for some i. Let the even-connections be, $x_{i-1}p_1p_2x_i$ and $x_ip_2p_1x_{i+1}$. Note that, in this case, for $r = 1, 2$, $p_r \neq x_j$ for any j and $\{p_r, x_j\} \notin E(G)$ for $j \neq i-1, i, i+1$. Therefore we have path $P' : x_{i-1}p_1x_{i+1}x_{i+2}$ in G. Since p_1 cannot be equal to x_j for $j < i-1$ and $j > i+2$, replace the segment $x_{i-1}x_ix_{i+1}x_{i+2}$ in P_k with P' to obtain an induced path $x_1 \cdots x_{i-2}P'x_{i+3} \cdots x_k$ with k vertices in G, which contradicts our hypothesis that G is P_k-free. Hence G' is P_k-free graph. $\square$

The following result compares the induced matching numbers of G and G'.

Proposition 4.4. *Let G be a graph and G' be the graph associated to $(\widetilde{I(G)^{s+1} : e_1 \cdots e_s})$ for $e_1, \ldots, e_s \in E(G)$. Then $\nu(G') \leq \nu(G)$.*

Proof. Let $\{f_1, \ldots, f_q, f_{q+1}, \ldots, f_r, f_{r+1}, \ldots, f_t\}$ be an induced matching of G', where

(1) for $\ell = 1, \ldots, q$, $f_\ell \in E(G)$;
(2) for $\ell = q+1, \ldots, r$, $f_\ell = \{u_\ell, v_\ell\}$ and $u_\ell \neq v_\ell$ are vertices of G even-connected with respect to $e_1 \cdots e_s$.
(3) for $\ell = r+1, \ldots, t$, $f_\ell = \{u_\ell, u'_\ell\}$, and u_ℓ is even-connected to itself with respect to $e_1 \cdots e_s$ and u'_ℓ is new vertex.

Let $u_\ell = p_0^\ell p_1^\ell \cdots p_{2k_\ell+1}^\ell = v_\ell$, for $\ell = q+1, \ldots, t$, be an even-connection between u_ℓ and v_ℓ (u_ℓ may be equal to v_ℓ) with respect to $e_1 \cdots e_s$. For $i = q+1, \ldots, t$, let $f'_i = \{p_0^i, p_1^i\}$.

Claim: $\{f_1, \ldots, f_q, f'_{q+1}, \ldots, f'_t\}$ is an induced matching for G.

Proof of the claim: Suppose this is not an induced matching. Then, either there is a common vertex for two of the edges or there exists an edge in G connecting two of the edges in the above set. Since $\{f_1, \ldots, f_t\}$ is an induced matching, we can see that f_i and f'_j cannot have a common vertex, for any $1 \leq i \leq q$, $q+1 \leq j \leq t$. Suppose $f'_\ell = \{p_0^\ell, p_1^\ell\}$ and $f'_m = \{p_0^m, p_1^m\}$ have a common vertex. If $p_0^\ell = p_0^m$ or $p_0^\ell = p_1^m$ or $p_1^\ell = p_0^m$, then this contradicts the assumption that $\{f_\ell, f_m\}$ form an induced matching. If $p_1^\ell = p_1^m$, then there is an even-connection

$$p_0^m(p_1^m = p_1^\ell)p_2^\ell \cdots p_{2k_\ell+1}^\ell$$

between p_0^m and $p_{2k_\ell+1}$ with respect to $e_1 \cdots e_s$, which contradicts the assumption that $\{f_\ell, f_m\}$ form an induced matching. Therefore f'_ℓ and f'_m cannot have a common vertex. Also, since $\{f_1, \ldots, f_q\}$ is an induced matching in G, there cannot be an edge connecting f_i and f_j. Therefore the two possibilities are,

(1) there exists an edge connecting f_i and f'_j;

(2) there exists an edge connecting f'_ℓ and f'_m.

Case 1: Let f_i and $f'_j = \{p_0^j, p_1^j\}$, for some $1 \le i \le q$ and $q+1 \le j \le t$, be connected by an edge, say $\{x_\alpha, x_\beta\}$ in G. If either $x_\alpha = p_0^j$ or $x_\beta = p_0^j$, then this is a contradiction to the assumption that f_i and f_j form an induced matching in G'. Therefore either $x_\alpha = p_1^j$ or $x_\beta = p_1^j$. Suppose $x_\alpha = p_1^j$. Then there is an even-connection $x_\beta p_1^j \cdots p_{2k_j+1}^j$ in G. This is also in contradiction to the assumption that f_i and f_j is an induced matching in G'. Therefore $x_\alpha \neq p_1^j$. Similarly one can prove that $x_\beta \neq p_1^j$. Therefore, there cannot be any common edge $\{x_\alpha, x_\beta\}$ between f_i and f'_j for any $j = q+1, \dots, t$.

Case 2: Suppose there exists an edge, say $\{x_\alpha, x_\beta\}$, between $f'_\ell = \{p_0^\ell, p_1^\ell\}$ and $f'_m = \{p_0^m, p_1^m\}$.

(1) If $\{x_\alpha, x_\beta\} = \{p_0^\ell, p_0^m\}$, then $\{x_\alpha, x_\beta\}$ is a common edge of f_ℓ and f_m, which is a contradiction to the assumption that $\{f_\ell, f_m\}$ is an induced matching in G'.

(2) If $\{x_\alpha, x_\beta\} = \{p_0^\ell, p_1^m\}$, then there is an even-connection
$$p_0^\ell p_1^m p_2^m \cdots p_{2k_m+1}^m$$
between p_0^ℓ and $p_{2k_m+1}^m$ with respect to $e_1 \cdots e_s$, which is again a contradiction. Similarly, one arrives at a contradiction if $\{x_\alpha, x_\beta\} = \{p_1^\ell, p_0^m\}$.

(3) Suppose $\{x_\alpha, x_\beta\} = \{p_1^\ell, p_1^m\}$. If $\{p_{2\mu+1}^\ell, p_{2\mu+2}^\ell\}$ and $\{p_{2\mu_1+1}^m, p_{2\mu_1+2}^m\}$ have a common vertex, for some $0 \le \mu \le k_\ell - 1$, $0 \le \mu_1 \le k_m - 1$, then by [3, Lemma 6.13], p_0^ℓ is even-connected either to p_0^m or to $p_{2k_m+1}^m$. Both contradicts the assumption that f_ℓ, f_m is an induced matching in G'. Therefore $\{p_{2\mu+1}^\ell, p_{2\mu+2}^\ell\}$ and $\{p_{2\mu_1+1}^m, p_{2\mu_1+2}^m\}$ does not have a common vertex for any $0 \le \mu \le k_\ell - 1$, $0 \le \mu_1 \le k_m - 1$. Then there is an even-connection
$$p_{2k_\ell+1}^\ell p_{2k_\ell}^\ell \cdots p_1^\ell p_1^m \cdots p_{2k_m+1}^m$$
between $p_{2k_\ell+1}^\ell$ and $p_{2k_m+1}^m$, which is also a contradiction to the assumption that $\{f_\ell, f_m\}$ is an induced matching.

Therefore $\{f_1, \dots, f_q, f'_{q+1}, \dots, f'_t\}$ form an induced matching of G. Hence $\nu(G') \le \nu(G)$. $\square$

As a consequence of Proposition 4.4, we obtain an upper bound for $\operatorname{reg}(I(G)^s)$ when G is a bipartite graph.

Corollary 4.5. *Let G be a bipartite graph with partitions X and Y. Then for all $s \ge 1$,*
$$\operatorname{reg}(I(G)^s) \le 2s + \frac{1}{2}\left(\nu(G) + \min\{|X|, |Y|\}\right) - 1.$$

Proof. If $s = 1$, then this is proved in [7, Theorem 4.16]. Let G' be the graph associated to $(I(G)^{s+1} : e_1 \cdots e_s)$ for some edges $e_1, \dots, e_s$ in G. Since $\nu(G') \le \nu(G)$ and G' is also bipartite with partitions X and Y, the assertion now follows from Theorem 2.4 and induction. $\square$

By [5, Theorem 4.5] and Corollary 4.5, for any bipartite graph G, we have
$$2s + \nu(G) - 1 \le \operatorname{reg}(I(G)^s) \le 2s + \frac{1}{2}\left(\nu(G) + \min\{|X|, |Y|\}\right) - 1 \text{ for any } s \ge 1. \tag{4.1}$$

Let G be a bipartite graph with partitions X and Y, say $|X| \le |Y|$. Then by Corollary 4.5,
$$\operatorname{reg}(I(G)^s) \le 2s + |X| - 1 \text{ for any } s \ge 1.$$

Remark 4.6. *It is easy to see that for a bipartite graph G, $\nu(G) \leq \min\{|X|, |Y|\}$. At the same time, the difference between $\nu(G)$ and $\min\{|X|, |Y|\}$ could be arbitrarily large, for example in the case of complete bipartite graphs. If $\nu(G) = \min\{|X|, |Y|\}$ or $\nu(G) = \min\{|X|, |Y|\} - 1$, then from (4.1), it follows that $\operatorname{reg}(I(G)^s) = 2s + \nu(G) - 1$. For example, let H be a bipartite graph with $V(H) = X \cup Y$. Let $X = \{x_1, \ldots, x_n\}$ and $n \leq |Y|$. Let G be the graph obtained by attaching at least one pendant vertex to each x_i's in H. Then $\nu(G) = n$ and hence $\operatorname{reg}(I(G)^s) = 2s + n - 1$.*

Another immediate consequence of the comparison between the induced matching numbers is the relation between the nK_2-free property of G and G'.

Corollary 4.7. *If G is an nK_2-free graph for some $n \geq 1$, then for any s-fold product $e_1 \cdots e_s$, the graph G' associated to $(I(G)^{s+1} : e_1 \cdots e_s)$ is nK_2-free.*

Proof. If G is nK_2-free, then $\nu(G) < n$. By Proposition 4.4, $\nu(G') \leq \nu(G)$. Hence $\nu(G') < n$ and hence G' is nK_2-free. □

Taking $n = 2$ in the previous corollary, we obtain [3, Lemma 6.14]. Using this result and [26, Proposition 1.3] we get:

Corollary 4.8. *If G is gap-free, then so is the graph G' associated to $(I(G)^{s+1} : e_1 \cdots e_s)$, for every s-fold product $e_1 \cdots e_s$. In other words, if $I(G)$ has linear presentation, then so has $(I(G)^{s+1} : e_1 \cdots e_s)$, for every s-fold product $e_1 \cdots e_s$.*

In [2], Banerjee posed questions on the relation between $I(G)$ and $(I(G)^{s+1} : e_1 \cdots e_s)$ for some edges $e_1, \ldots, e_s$ in G. In particular, he asked

Question 4.9. [2, Question 6.2.2] *Classify G and $e_1, \ldots, e_s$ such that $\operatorname{reg}(I(G)) \geq \operatorname{reg}(I(G)^{s+1} : e_1 \cdots e_s)$.*

As an application of our results Corollary 3.4 and Proposition 4.4, we obtain some sufficient conditions for which the above inequality holds true.

Proposition 4.10. *Let G be any graph and $e_1, \ldots, e_s$ be edges of G, for some $s \geq 1$. Let G' be the graph associated to $(I(G)^{s+1} : e_1 \cdots e_s)$. Then the inequality $\operatorname{reg}(I(G)) \geq \operatorname{reg}(I(G)^{s+1} : e_1 \cdots e_s)$ holds true if*

(1) $\operatorname{reg}(I(G)^{s+1} : e_1 \cdots e_s) = \nu(G') + 1$; *or*
(2) *G is bipartite with $\operatorname{reg}(I(G)) = \text{co-chord}(G) + 1$.*

Proof. If $\operatorname{reg}(I(G)^{s+1} : e_1 \cdots e_s) = \nu(G') + 1$, then

$$\begin{array}{rcll} \operatorname{reg}(I(G)^{s+1} : e_1 \cdots e_s) & = & \nu(G') + 1 & \\ & \leq & \nu(G) + 1 & \text{(by Proposition 4.4)} \\ & = & \operatorname{reg}(I(G)). & \text{(by (1.1))} \end{array}$$

If G is bipartite and $\operatorname{reg}(I(G)) = \text{co-chord}(G) + 1$, then

$$\begin{array}{rcll} \operatorname{reg}(I(G)^{s+1} : e_1 \cdots e_s) & \leq & \text{co-chord}(G') + 1 & \text{(by (1.1))} \\ & \leq & \text{co-chord}(G) + 1 & \text{(by Corollary 3.4)} \\ & = & \operatorname{reg}(I(G)). & \end{array}$$

□

The *chromatic number* denoted by $\chi(G)$ is the smallest number of colors possible in a proper vertex coloring of G, see [31, Definition 5.1.4]. Note that, if G is bipartite graph, then $\alpha(G) = \chi(G^c)$, see [31, Chapter 8]. We now recall some results from the literature:

Remark 4.11. *Let G be a graph.*

(1) *If G is unmixed bipartite, then Woodroofe proved that* $\nu(G) = \text{co-chord}(G)$, [32, Theorem 16].
(2) *If G is a graph, then Woodroofe proved that* $\text{co-chord}(W(G)) = \chi(G^c)$ *and* $\alpha(G) = \nu(W(G))$, [32, Lemma 21].
(3) *If G is a weakly chordal graph, then Busch-Dragan-Sritharan proved that* $\nu(G) = \text{co-chord}(G)$, [8, Proposition 3].

As an immediate consequence, we have

Corollary 4.12. *Let G be a bipartite graph and $e_1, \ldots, e_s$ be edges of G. Then* $\text{reg}(I(G)^{s+1} : e_1 \cdots e_s) \leq \text{reg}(I(G))$ *if*

(1) *G is unmixed;*
(2) *G is weakly chordal;*
(3) *$G = W(H)$ for some bipartite graph H or*
(4) *G is P_6-free;*

Proof. If G is unmixed, weakly chordal or $G = W(H)$ for some bipartite graph H, then the assertion follows from Remark 4.11 and Proposition 4.10. If G is P_6-free, then by Theorem 4.3 G' is also P_6-free. By [6, Theorem 3.15], $\text{reg}(I(G')) = \nu(G') + 1$. Now the assertion follows from Proposition 4.10. $\square$

Below, we present an example to show that the inequalities in Corollary 3.4, Proposition 4.4 and Proposition 4.10 could be strict.

Example 4.13. *Let $G = C_6$ and $I = I(G) = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_1) \subset k[x_1, \ldots, x_6]$. Then $(I^3 : x_2x_3x_4x_5) = I + (x_1x_4, x_3x_6)$. Let G' be the graph associated to $(I^3 : x_2x_3x_4x_5)$. It can be easily seen that* $\text{co-chord}(G) = \nu(G) = 2$ *and* $\text{co-chord}(G') = \nu(G') = 1$. *By (1.1),* $\text{reg}(I) = 3$ *and* $\text{reg}(I^3 : x_2x_3x_4x_5) = 2$.

5. Precise expressions for asymptotic regularity

In this section, we apply Theorem 3.6 to obtain precise expressions for the regularity of powers of edge ideals of various subclasses of bipartite graphs. We begin the study with some classes of graphs G for which $\nu(G) = \text{co-chord}(G)$. We then use (3.1) to prove that $\text{reg}(I(G)^s) = 2s + \nu(G) - 1$ for such graphs.

Corollary 5.1. *Let G be a bipartite graph. If*

(1) *G is unmixed;* [2]
(2) *$G = W(H)$ for some bipartite graph H;*
(3) *G is weakly chordal, or*
(4) *If G is P_6-free graph,*

then for all $s \geq 1$, $\text{reg}(I(G)^s) = 2s + \nu(G) - 1$.

[2] In a personal communication, we have been informed that Banerjee and Mukundan have also obtained Corollary 5.1(1).

Proof. The assertions, (1), (2) and (3) follows from Remark 4.11 and (3.1). If G is P_6-free, then so is G'. Therefore, the result now follows from Theorem 2.4 and Corollary 4.12. $\square$

Observe that bipartite P_5-free graphs are chordal bipartite. Therefore by Corollary 5.1(3), $\operatorname{reg}(I(G)^s) = 2s + \nu(G) - 1$ for all $s \geq 1$. In general, for a bipartite P_6-free graph, it is not known whether the equality $\nu(G) = \text{co-chord}(G)$ is true. However, the previous result shows that for $s \geq 1$, $\operatorname{reg}(I(G))^s) = 2s + \nu(G) - 1$.

Since forests are weakly chordal bipartite graphs, we derive, from Corollary 5.1(3), one of the main results of Beyarslan, Hà and Trung:

Corollary 5.2. [5, Theorem 4.7] *If G is a forest, then for all $s \geq 1$,*

$$\operatorname{reg}(I(G)^s) = 2s + \nu(G) - 1.$$

The *bipartite complement* of a bipartite graph G is the bipartite graph G^{bc} on the same vertex set as G, $V(G^{bc}) = X \cup Y$, with $E(G^{bc}) = \{\{x, y\} | x \in X, y \in Y, \{x, y\} \notin E(G)\}$. Below we make an observation on connected bipartite graphs G with $\operatorname{reg}(I(G)) = 3$.

Observation 5.3. *If G is a connected bipartite graph with* $\operatorname{reg}(I(G)) = 3$*, then by* [11, Theorem 3.1]*,* $2 \leq \nu(G)$*. If* $2 < \mathrm{b}(G)$*, then G^{bc} has an induced cycle of length* 6*, which contradicts* [11, Theorem 3.1]*. Therefore,* $\nu(G) = \text{co-chord}(G) = \mathrm{b}(G) = 2$*.*

We now derive two results of Alilooee and Banerjee ([1, Theorems 3.8, 3.9]) as a corollary:

Corollary 5.4. *If G is a connected bipartite graph with* $\operatorname{reg}(I(G)) = 3$*, then*

(1) $\operatorname{reg}(I(G)^{s+1} : e_1 \cdots e_s) \leq 3$ *for any s-fold product $e_1 \cdots e_s$;*
(2) *for all* $s \geq 1$, $\operatorname{reg}(I(G)^s) = 2s + 1$.

Proof. The first assertion follows from Observation 5.3 and Proposition 4.10 and then second assertion follows from Observation 5.3 and (3.2). $\square$

So far, we had been discussing about graphs G for which $\operatorname{reg}(I(G)^s) = 2s + \nu(G) - 1$ for all $s \geq 1$. Now we produce some classes of graphs G for which $\text{co-chord}(G) - \nu(G)$ is arbitrarily large and hence $\operatorname{reg}(I(G)^s) - [2s + \nu(G) - 1]$ is also arbitrarily large. If G is the disjoint union of $C_{n_1}, \ldots, C_{n_m}$ and k edges, then one can easily see that

$$\nu(G) = k + \sum_{j=1}^{m} \left\lfloor \frac{n_j}{3} \right\rfloor,$$

$$\text{co-chord}(G) = \begin{cases} k + \sum_{j=1}^{m} \lfloor \frac{n_j}{3} \rfloor & \text{if } n_1, \ldots, n_m \equiv \{0, 1\} \ (mod\ 3), \\ k + m + \sum_{j=1}^{m} \lfloor \frac{n_j}{3} \rfloor & \text{if } n_1, \ldots, n_m \equiv 2 \ (mod\ 3), \end{cases}$$

and

$$\operatorname{reg}(I(G)) = \begin{cases} \nu(G) + 1 & \text{if } n_1, \ldots, n_m \equiv \{0, 1\} \ (mod\ 3), \\ \text{co-chord}(G) + 1 & \text{if } n_1, \ldots, n_m \equiv 2 \ (mod\ 3). \end{cases}$$

We prove a precise expression for $\operatorname{reg}(I(G)^s)$ in this case. We would like to thank Tai Hà for indicating the following proof, much simpler than the original one.

Theorem 5.5. *Let G be the disjoint union of $C_{n_1}, \dots, C_{n_m}$ and k edges, $k \geq 1$. Then for all $s \geq 1$,*

$$\operatorname{reg}(I(G)^s) = \begin{cases} 2s + \nu(G) - 1 & \text{if } n_1, \dots, n_m \equiv \{0,1\} \ (mod\ 3) \\ 2s + \text{co-chord}(G) - 1 & \text{if } n_1, \dots, n_m \equiv 2 \ (mod\ 3) \end{cases}$$

Proof. Let $e_1, \dots, e_k$ be disjoint edges. It follows from [17, Theorem 7.6.28] and [5, Theorem 5.2] that

(1) if $n \equiv \{0,1\}(mod\ 3)$, then $\operatorname{reg}(I(C_n)^s) = 2s + \nu(C_n) - 1$, for all $s \geq 1$;
(2) if $n \equiv 2(mod\ 3)$, then $\operatorname{reg}(I(C_n)) = \text{co-chord}(C_n)+1$ and $\operatorname{reg}(I(C_n)^s) = 2s+\nu(C_n)-1$, for all $s \geq 2$.

Let $G_1 = C_{n_1} \cup \{e_1\} \cup \dots \cup \{e_k\}$. First we claim that, for $s \geq 1$

$$\operatorname{reg}(I(G_1)^s) = \begin{cases} 2s + \nu(G_1) - 1 & \text{if } n_1 \equiv \{0,1\} \ (mod\ 3) \\ 2s + \text{co-chord}(G_1) - 1 & \text{if } n_1 \equiv 2 \ (mod\ 3) \end{cases}$$

We prove this by induction on k. Let $k = 1$. Let $H_1 = C_{n_1} \cup \{e_1\}$. By [14, Lemma 2.5], we can prove the case $s = 1$. If $n_1 \equiv \{0,1\}(mod\ 3)$, then by [27, Theorem 5.7], for $s \geq 2$, we have

$$\operatorname{reg}(I(H_1)^s) = 2s + \nu(H_1) - 1.$$

If $n_1 \equiv 2(mod\ 3)$, then by [14, Proposition 2.7], we get

$$\operatorname{reg}(I(H_1)^2) = \operatorname{reg}(I(C_n)) + \operatorname{reg}(I(e_1)^2) - 1 = \text{co-chord}(H_1) + 3.$$

By [27, Theorem 5.7], for $s \geq 3$, we have

$$\operatorname{reg}(I(H_1)^s) = 2s + \text{co-chord}(H_1) - 1.$$

This completes the proof for $k = 1$. Suppose $k > 1$. Let $G_1 = C_{n_1} \cup \{e_1, \dots, e_k\}$, where $e_1, \dots, e_k$ are disjoint edges. Let $H = C_{n_1} \cup \{e_1, \dots, e_{k-1}\}$. By induction hypothesis, for $s \geq 1$,

$$\operatorname{reg}(I(H)^s) = \begin{cases} 2s + \nu(H) - 1 & \text{if } n_1 \equiv \{0,1\}(mod\ 3), \\ 2s + \text{co-chord}(H) - 1 & \text{if } n_1 \equiv 2(mod\ 3). \end{cases}$$

Since $G_1 = H \cup \{e_k\}$. By [14, Lemma 2.5] and [27, Theorem 5.7], for $s \geq 1$, we have

$$\operatorname{reg}(I(G_1)^s) = \begin{cases} 2s + \nu(G_1) - 1 & \text{if } n_1 \equiv \{0,1\}(mod\ 3), \\ 2s + \text{co-chord}(G_1) - 1 & \text{if } n_1 \equiv 2(mod\ 3). \end{cases}$$

Let $G_{m-1} = C_{n_1} \cup \dots \cup C_{n_{m-1}} \cup \{e_1, \dots, e_k\}$. Then by induction on m, we get, for $s \geq 1$,

$$\operatorname{reg}(I(G_{m-1})^s) = \begin{cases} 2s + \nu(G_{m-1}) - 1 & \text{if } n_1, \dots, n_{m-1} \equiv \{0,1\}(mod\ 3), \\ 2s + \text{co-chord}(G_{m-1}) - 1 & \text{if } n_1, \dots, n_{m-1} \equiv 2(mod\ 3). \end{cases}$$

Let $G = G_{m-1} \cup C_{n_m}$. By [14, Lemma 2.5], [14, Proposition 2.7] and [27, Theorem 5.7], we have $s \geq 1$,

$$\operatorname{reg}(I(G)^s) = \begin{cases} 2s + \nu(G) - 1 & \text{if } n_1, \dots, n_m \equiv \{0,1\}(mod\ 3), \\ 2s + \text{co-chord}(G) - 1 & \text{if } n_1, \dots, n_m \equiv 2(mod\ 3). \end{cases}$$

This completes the proof of the theorem. $\square$

It may be noted that C_n is not bipartite if $n = 2k+1$ for some k, but the upper bound in Theorem 3.6 is still satisfied in this case. Suppose $H \cong \coprod_{j=1}^{m} C_{3n_j+2} \coprod \coprod_{i=1}^{k} e_i$, then $\nu(H) = k + \sum_{j=1}^{m} n_j$ and co-chord$(H) = k + m + \sum_{j=1}^{m} n_j$. By Theorem 5.5, for $s \geq 1$, we have

$$\operatorname{reg}(I(H)^s) - [2s + \nu(G) - 1] = m.$$

Woodroofe proved that if H is an induced subgraph of a graph G, then $\operatorname{reg}(I(G)) \geq k+m+\sum_{j=1}^{m} n_j + 1$, [32, Corollary 11]. We obtain a similar bound for all the powers.

Corollary 5.6. *If a graph G has an induced subgraph $H \cong \coprod_{j=1}^{m} C_{n_j} \coprod \coprod_{i=1}^{k} e_i$, then*

$$\operatorname{reg}(I(G)^s) \geq \begin{cases} 2s + (k + \sum_{j=1}^{m} \lfloor \frac{n_j}{3} \rfloor) - 1 & \text{if } n_1, \ldots, n_m \equiv \{0,1\} \ (mod\ 3), \\ 2s + (k + m + \sum_{j=1}^{m} \lfloor \frac{n_j}{3} \rfloor) - 1 & \text{if } n_1, \ldots, n_m \equiv 2 \ (mod\ 3). \end{cases}$$

Proof. Follows immediately from Theorem 5.5 and [5, Corollary 4.3]. □

Note that if $n_j = 3k_j + 2$ for some $k_j \geq 1$, $j = 1, \ldots, m$, then $\operatorname{reg}(I(G)^s) \geq 2s +$ co-chord$(H) - 1$. This is a much improved lower bound in this class of graphs since co-chord(H) could be much larger than $\nu(G)$.

Example 5.7. *Let $K_{1,n}$ be the complete bipartite graph with partition $\{w\} \cup \{x_1, \ldots, x_n\}$. Let $n = k+m$. Let G be the graph obtained by attaching a pendant vertex each to $x_1, \ldots, x_k$ and identifying a vertex of C_{2r_t} with x_t for $k+1 \leq t \leq n$, where $2r_t \equiv 2 (mod\ 3)$. Let H be induced subgraph of G on $V(G) \setminus \{w\}$. One can easily see that,* co-chord$(H) =$ co-chord(G). *Therefore it follows from Corollary 5.6 and Theorem 3.6 that for all $s \geq 1$,*

$$\operatorname{reg}(I(G)^s) = 2s + \text{co-chord}(G) - 1.$$

It may also be noted that for this class of graphs $\operatorname{reg}(I(G)^s) - [2s + \nu(G) - 1] = m$.

There are many classes of graphs G for which the equality $\operatorname{reg}(I(G)^s) = 2s + b$ has been established, where b is some combinatorial invariant associated with G. For all such results, the constant term b is either equal to $\nu(G) - 1$ or equal to co-chord$(G) - 1$. While $\nu(G) - 1$ is a lower bound for $\operatorname{reg}(I(G)^s) - 2s$ for all graphs, co-chord$(G) - 1$ is an upper bound in the case of bipartite graphs. Moreover, co-chord$(G) - \nu(G)$ can be arbitrarily large. We conclude our article with the following question:

Question 5.8. *Does there exist a graph G such that for all $s \gg 0$*

$$2s + \nu(G) - 1 < \operatorname{reg}(I(G)^s) < 2s + \text{co-chord}(G) - 1?$$

More generally, let $n =$ co-chord$(G) - \nu(G)$ and $\mathcal{C}_n = \{G \mid$ co-chord$(G) - \nu(G) = n\}$. For each $t \in \{0, 1, \ldots, n\}$, does there exist $G_t \in \mathcal{C}_n$ such that $\operatorname{reg}(I(G_t)^s) = 2s + \nu(G) + t - 1$?

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E-mail address: `jayanav@iitm.ac.in`

E-mail address: `naru@iitm.ac.in`

E-mail address: `selva.y2s@gmail.com`

Department of Mathematics, Indian Institute of Technology Madras, Chennai, INDIA - 60036