# REGULARITY OF BINOMIAL EDGE IDEALS OF CERTAIN BLOCK GRAPHS 

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#### Abstract

We prove that the regularity of binomial edge ideals of graphs obtained by gluing two graphs at a free vertex is the sum of the regularity of individual graphs. As a consequence, we generalize certain results of Zafar and Zahid. We obtain an improved lower bound for the regularity of trees. Further, we characterize trees which attain the lower bound. We prove an upper bound for the regularity of certain subclass of blockgraphs. As a consequence we obtain sharp upper and lower bounds for a class of trees called lobsters.


## 1. Introduction

Let $G$ be a simple graph on the vertex set $[n]$. Let $S=K\left[x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n}\right]$ be the polynomial ring in $2 n$ variables, where $K$ is a field. Then the ideal $J_{G}$ generated by $\left\{x_{i} y_{j}-x_{j} y_{i} \mid(i, j)\right.$ is an edge in $\left.G\right\}$ is called the binomial edge ideal of $G$. This was introduced by Herzog et al., [8] and independently by Ohtani, [12]. Recently, there have been many results relating the combinatorial data of graphs with the algebraic properties of the corresponding binomial edge ideals, see [1], [2, 4], [11], [14], [15], [17]. In particular, there have been active research connecting algebraic invariants of the binomial edge ideals such as Castelnuovo-Mumford regularity, depth, Betti numbers etc., with combinatorial invariants associated with graphs such as length of maximal induced path, number of maximal cliques, matching number. For example, Matsuda and Murai proved that $\ell \leq \operatorname{reg}\left(S / J_{G}\right) \leq n-1$, where $\ell$ is the length of the longest induced path in $G$, [11]. They conjectured that if $\operatorname{reg}\left(S / J_{G}\right)=n-1$, then $G$ is a path of length $n$. In [10], Kiani and Saeedi Madani proved the conjecture. Chaudhry et al. proved that if $T$ is a tree whose longest induced path has length $\ell$, then $\operatorname{reg}\left(S / J_{T}\right)=\ell$ if and only if $T$ is a caterpillar, [1]. Therefore, the trees that attain the minimal or maximal regularity have been characterized. However, for most of the graph classes, the Matsuda-Murai bounds are far from being tight. Saeedi Madani and Kiani, [14], proved that if $G$ is a closed graph, then $\operatorname{reg}\left(S / J_{G}\right) \leq c(G)$, where $c(G)$ is the number of maximal cliques in $G$. Here, a graph is said to be closed if its binomial edge ideal has a quadratic Gröbner basis. They generalized this result to the case of binomial edge ideal of a pair of a closed graph and a complete graph, and proposed conjectured that for any graph $G, \operatorname{reg}\left(S / J_{G}\right) \leq c(G)$, [15]. In [9], they proved the conjecture for generalized block graph. In [5], Ene and Zarojanu proved that if $G$ is a chordal graph with the property that any two distinct maximal cliques intersect in at most one vertex, then $\operatorname{reg}\left(S / J_{G}\right) \leq c(G)$.

Though the bound obtained for block graphs by Madani and Kiani is sharp, there are several subclasses of block graphs, including trees, where the upper bound is more than the actual regularity (for example, caterpillar, [1]). In this article, we study the regularity of binomial edge ideals of certain classes of block graphs, and in particular trees.

In [13], Rauf and Rinaldo studied binomial edge ideals of graphs obtained by gluing two graphs at free vertices. We extend their arguments to observe that the regularity of the binomial edge ideal of a graph obtained by gluing two graphs at free vertices is equal to the sum of the regularities the binomial edge ideals of the individual graphs, Theorem 3.1. As a consequence, we obtain precise expressions for the regularities of several classes of trees and block graphs, (Corollaries 3.2, 3.3 and 3.4).

The lower bound for the regularity of a binomial edge ideal given by Matsuda and Murai, namely, the length of the longest induced path, [11, is the best lower bound known as of now. By using inductive application of Theorem 3.1, we obtain a lower bound for the regularity of binomial edge ideals of trees in terms of the number of internal vertices, Theorem 4.1. We characterize trees which attain the lower bound in terms of presence of a specific tree as a subgraph, Theorem 4.2.

We then move on to study certain subclasses of block graphs and obtain improved upper bounds for their regularity, Theorems 4.4 and 4.5. As a consequence we get an upper bound for the regularity of the binomial edge ideals of lobsters (see Section 2 for definition), Corollary 4.6. We also obtain a precise expression for the regularity of binomial edge ideals of a subclass of lobsters, called pure lobsters, in Corollary 4.3.

## 2. Preliminaries

In this section, we set up the basic definitions and notation.
Let $G$ be a finite simple graph. A vertex $x$ of $G$ is said to be a cut vertex if $G \backslash\{x\}$ has strictly more connected components than $G$. A block of $G$ is a maximal subgraph without a cut vertex. A graph $G$ is a block graph if every block of $G$ is a complete graph.

Let $T$ be a tree and $\mathrm{L}(T)=\{v \in V(T) \mid \operatorname{deg}(v)=1\}$ be the set of all leaves of $T$. We say that a tree $T$ is a caterpillar if $T \backslash \mathrm{~L}(T)$ is either empty or is a simple path. Similarly, a tree $T$ is said to be a lobster, if $T \backslash \mathrm{~L}(T)$ is a caterpillar, [6]. Observe that every caterpillar is also a lobster. A longest path in a lobster is called a spine of the lobster. Note that given any spine, every edge of a caterpillar is incident to it. With respect to a fixed spine $P$, the pendant edges incident with $P$ are called whiskers. It can be seen that every non-leaf vertex $u$ not incident on a fixed spine $P$ of a lobster forms the center of a star $\left(K_{1, m}, m \geq 2\right)$. Each such star is said to be a limb with respect to $P$. More generally, given a vertex $v$ on any simple path $P$, we can attach a star $\left(K_{1, m}, m \geq 2\right)$ with center $u$ by identifying exactly one of the leaves of the star with $v$. Such a star is called a limb attached to $P$.

Note that the limbs and whiskers depend on the spine. Whenever a spine is fixed, we will refer to them simply as limb and whisker.

Example 2.1. Let $G$ denote the given graph on 10 vertices:

$G \quad$ induced by $\{2,3,4\}$ is a limb.

We now describe the construction of a useful exact sequence introduced by Ene, Herzog and Hibi, 4].
2.1. Ene-Herzog-Hibi Process. Let $G$ be a block graph, $\Delta(G)$ be the clique complex of $G$ and $F_{1}, \ldots, F_{r}$ be a leaf order on the facets of $\Delta(G)$. Assume that $r>1$. Let $v \in V(G)$ be the unique vertex in $F_{r}$ such that $F_{r} \cap F_{j} \subseteq\{v\}$ for all $j<r$. Let $G^{\prime}$ be the graph obtained by adding necessary edges to $G$ so that $N(v) \cup\{v\}$ is a clique. Let $G^{\prime \prime}$ be the graph induced on $G \backslash\{v\}$ and $H$ be the graph induced on $G^{\prime} \backslash\{v\}$. Then there exists an exact sequence

$$
\begin{equation*}
0 \rightarrow S / J_{G} \rightarrow S / J_{G^{\prime}} \oplus S / J_{G^{\prime \prime}} \rightarrow S / J_{H} \rightarrow 0 \tag{1}
\end{equation*}
$$

We call $G^{\prime}, G^{\prime \prime}$ and $H$ to be the graphs obtained by applying EHH-process on $G$ with respect to $v$. This exact sequence has been found extremely useful in inductive arguments in the study of homological properties of the binomial edge ideals.

## 3. Regularity via gluing

In this section, we describe the process of gluing and use it to obtain precise regularity expressions for certain classes of graphs. Let $G$ be a graph and $v$ be a cut vertex in $G$. Let $G_{1}, \ldots, G_{k}$ be the components of $G \backslash\{v\}$ and $G_{i}^{\prime}=G\left[V\left(G_{i}\right) \cup\{v\}\right]$, the subgraph of $G$ induced by $V\left(G_{i}\right) \cup\{v\}$. Then, $G_{1}^{\prime}, \ldots, G_{k}^{\prime}$ is called the split of $G$ at $v$ and we say that $G$ is obtained by gluing $G_{1}, \ldots, G_{k}$ at $v$.

Theorem 3.1. Let $G_{1}$ and $G_{2}$ be the split of a graph $G$ at $v$. If $v$ is a free vertex in both $G_{1}$ and $G_{2}$, then

$$
\operatorname{reg}\left(S / J_{G}\right)=\operatorname{reg}\left(S / J_{G_{1}}\right)+\operatorname{reg}\left(S / J_{G_{2}}\right)
$$

Proof. Let $G_{1}$ and $G_{2}$ be graphs on the vertices $\{1, \ldots, n\}$ and $\{n+1, \ldots, n+m\}$ respectively. Assume that $n$ is a free vertex in $G_{1}$ and $n+m$ is a free vertex in $G_{2}$. Let $G$ be the graph obtained by identifying vertices $n$ and $n+m$ in $G_{1} \cup G_{2}$, i.e., $v=n=n+m$. Let $G^{\prime}=G_{1} \cup G_{2}$ and $S^{\prime}=K\left[x_{1}, \ldots, x_{n+m}, y_{1}, \ldots, y_{n+m}\right]$. Then it can be easily seen that $S / J_{G} \cong S^{\prime} /\left(J_{G^{\prime}}+\left(x_{n}-x_{n+m}, y_{n}-y_{n+m}\right)\right)$. From the proof of Theorem 2.7 in [13], it follows that $\left(x_{n}-x_{n+m}, y_{n}-y_{n+m}\right)$ is a regular sequence on $S^{\prime} / J_{G^{\prime}}$. Hence the assertion follows.

As an immediate consequence, we have the following:
Corollary 3.2. Let $G=G_{1} \cup \cdots \cup G_{k}$ be such that
(1) for $i \neq j$, if $G_{i} \cap G_{j} \neq \emptyset$, then $G_{i} \cap G_{j}=\left\{v_{i j}\right\}$, for some vertex $v_{i j}$ which is a free vertex in $G_{i}$ as well as $G_{j}$;
(2) for distinct $i, j, k, G_{i} \cap G_{j} \cap G_{k}=\emptyset$.

Then $\operatorname{reg} S / J_{G}=\sum_{i=1}^{k} \operatorname{reg} S / J_{G_{i}}$.
Recall that for a (generalized) block graph $G$, $\operatorname{reg} S / J_{G} \leq c(G)$, 9]. We obtain a subclass of block graphs which attain this bound.

Corollary 3.3. If $G$ is a block graph such that no vertex is contained in more than two maximal cliques, then $\operatorname{reg} S / J_{G}=c(G)$.
Proof. We use induction on $c(G)$. If $c(G)=1$, then $G$ is a complete graph and hence $\operatorname{reg} S / J_{G}=1$. Now assume that $c(G)>1$. Consider any cut vertex $v$ of $G$. Let $G_{1}$ and $G_{2}$ be the split of $G$ at $\{v\}$. Then, $c(G)=c\left(G_{1}\right)+c\left(G_{2}\right)$. Now the result follows from Corollary 3.2 and induction hypothesis.

In [17], Zafar and Zahid considered special classes of graphs called $\mathcal{G}_{3}$ and $\mathcal{T}_{3}$ and obtained the regularities of the corresponding binomial edge ideals. We generalize their results:

Corollary 3.4. (1) Let $P_{1}, \ldots, P_{s}$ be paths of lengths $r_{1}, \ldots, r_{s}$ respectively. Let $G$ be the graph obtained by identifying a leaf of $P_{i}$ with the $i$-th vertex of the complete graph $K_{s}$. Then $\operatorname{reg}\left(S / J_{G}\right)=1+\sum_{i=1}^{s} r_{i}$.
(2) Let $P_{1}, \ldots, P_{k}$ be paths of lengths $r_{1}, \ldots, r_{k}$ respectively. Let $G$ be the graph obtained by identifying a leaf of $P_{i}$ with the $i$-th leaf of the star $K_{1, k}$. Then, $\operatorname{reg}\left(S / J_{G}\right)=2+\sum_{i=1}^{k} r_{i}$.
Proof. Both the assertions follow from Theorem 3.1.

## 4. Regularity of Block Graphs

In this section, we study the regularity of binomial edge ideals of certain block graphs, and in particular trees. We first obtain a lower bound for the regularity. We then consider a class of graphs, called lobsters, which are a generalization of caterpillars. We generalize a result of Chaudhry et al. to obtain sharp upper bounds for the regularity of binomial edge ideals of lobsters. It was shown by Matsuda and Murai, [11, Corollary 2.3], that for any graph $G, \operatorname{reg}\left(S / J_{G}\right) \geq \ell$, where $\ell$ is the length of the longest induced path in $G$. Below we prove a much improved lower bound, for the class of trees.

For a tree $T$, let $\operatorname{iv}(T):=\#\{$ internal vertices of $T\}$. Given a tree $T$, it is easy to see that one can construct $T$ from the trivial graph by adding vertices $v_{i}$ to $T_{i-1}$ at step $i$ to get $T_{i}$ so that $v_{i}$ is a leaf in the tree $T_{i}$. Any such ordering of vertices is called a leaf ordering.

Theorem 4.1. For a tree $T, \operatorname{reg}\left(S / J_{T}\right) \geq \operatorname{iv}(T)+1$.
Proof. Let $v_{1}, \ldots, v_{r}$ be a leaf ordering of the vertices of $G$, and let $G_{i}$ be the subgraph of $G$ induced by $v_{1}, \ldots, v_{i}$. Let $m_{i}$ denote the number of internal vertices of $G_{i}$. We argue by induction on $i$. If $i=2$, then $G_{2}$ is an edge and hence $\operatorname{reg}\left(S / J_{G_{2}}\right)=1$. Therefore,
the result holds. Assume the result for $G_{i}$. Then $G_{i+1}$ is obtained by adding a leaf $v_{i+1}$ to some vertex $v$ of $G_{i}$. If $v$ is a leaf in $G_{i}$, then $v$ is a free vertex in $G_{i}$, and hence by Theorem 3.1, $\operatorname{reg}\left(S / J_{G_{i+1}}\right)=\operatorname{reg}\left(S / J_{G_{i}}\right)+1$. Further, $v$ becomes a new internal vertex in $G_{i+1}$, i.e., $m_{i+1}=m_{i}+1$, and therefore the result holds. If $v$ is an internal vertex in $G_{i}$, then $m_{i+1}=m_{i}$ and since $G_{i}$ is an induced subgraph of $G_{i+1}, \operatorname{reg}\left(S / J_{G_{i+1}}\right) \geq$ $\operatorname{reg}\left(S / J_{G_{i}}\right) \geq m_{i}+1=m_{i+1}+1$ as required.


Let $T$ be the tree given on the left. It follows from [1, Theorem 4.1] and Theorem [3.1 that $\operatorname{reg}\left(S / J_{T}\right)=26=\operatorname{iv}(T)+1$, while the longest path of $T$ has length 15.

It is interesting to note that the graph $\mathcal{J}$, which we call Jewel, is the smallest tree for which $\operatorname{reg}\left(S / J_{\mathcal{J}}\right)>\operatorname{iv}(\mathcal{J})+1$. In fact, we can make the gap between the regularity and the number of internal vertices arbitrarily large by attaching edge disjoint copies of Jewel to leaves of any arbitrary tree. For example, Figure 3, which is two copies of the jewel superimposed together, has regularity 12 , much larger than the number of internal vertices which is 7 .

$\mathcal{J}:$ Jewel

We now characterize trees which attain the minimal regularity.
Theorem 4.2. A tree $T$ contains Jewel as a subgraph if and only if $\operatorname{reg}\left(S / J_{T}\right) \geq \operatorname{iv}(T)+2$.
Proof. Suppose $T$ is a tree on $[n]$ containing Jewel, $\mathcal{J}$, as a subgraph. Note that there is a leaf ordering $v_{1}, \ldots, v_{n}$ such that $V(\mathcal{J})=\left\{v_{1}, \ldots, v_{10}\right\}$. Recall that $\operatorname{reg}\left(S / J_{\mathcal{J}}\right)=$ $6=\operatorname{iv}(\mathcal{J})+2$. Let $G_{i}$ denote the subgraph of $T$ on the vetex set $\left\{v_{1}, \ldots, v_{i}\right\}, i \geq 10$. For each $i \geq 10, \operatorname{reg}\left(S / J_{G_{i+1}}\right)=\operatorname{reg}\left(S / J_{G_{i}}\right)+1$ if the neighbor of $v_{i+1}$ is a leaf in $G_{i}$ and $\operatorname{reg}\left(S / J_{G_{i+1}}\right) \geq \operatorname{reg}\left(S / J_{G_{i}}\right)$ otherwise. Note also that the neighbor of $v_{i+1}$ is a leaf in $G_{i}$ if and only if $\operatorname{iv}\left(G_{i+1}\right)=\operatorname{iv}\left(G_{i}\right)+1$. Since $\operatorname{reg}\left(S / J_{G_{10}}\right)=\operatorname{iv}\left(G_{10}\right)+2$, we get that $\operatorname{reg}\left(S / J_{G_{i}}\right) \geq \operatorname{iv}\left(G_{i}\right)+2$ for all $i \geq 10$. Hence the assertion follows.

Conversely, suppose $\operatorname{reg}\left(S / J_{T}\right) \geq \operatorname{iv}(T)+2$. First assume that $T$ does not have a vertex of degree 2. If $T$ is a caterpillar, then by [1], $\operatorname{reg}\left(S / J_{T}\right)=\operatorname{iv}(T)+1$. Therefore, $T$ is not a caterpillar. Then it contains the $Y$ graph ( $K_{1,3}$ attached with a leaf at each of its leaf vertices) as a subgraph [16, Theorem 2.2.19]. Since $T$ does not have vertices of degree 2, each degree 2 vertex in the $Y$ graph must have one more neighbour in $T$, which induces a Jewel in $T$.

Now assume that $T$ contains a vertex of degree 2 . Let $T^{\prime}$ be a minimal (with respect to the number of vertices) subgraph of $T$ so that $\operatorname{reg}\left(S / J_{T^{\prime}}\right) \geq \operatorname{iv}\left(T^{\prime}\right)+2$. If $T^{\prime}$ has no vertex of degree 2 , then $T^{\prime}$ contains a jewel. Suppose $T^{\prime}$ contains degree 2 vertex, say $v$. Let $T_{1}$ and $T_{2}$ be the split of $T^{\prime}$ at $\{v\}$. Note that $\operatorname{iv}\left(T^{\prime}\right)=\operatorname{iv}\left(T_{1}\right)+\operatorname{iv}\left(T_{2}\right)+1$. By Theorem 3.1, $\operatorname{reg}\left(S / J_{T^{\prime}}\right)=\operatorname{reg}\left(S / J_{T_{1}}\right)+\operatorname{reg}\left(S / J_{T_{2}}\right) \geq \operatorname{iv}\left(T_{1}\right)+\operatorname{iv}\left(T_{2}\right)+3$. Hence there exists $i \in\{1,2\}$ such that $\operatorname{reg}\left(S / J_{T_{i}}\right) \geq \operatorname{iv}\left(T_{i}\right)+2$. Since $T_{i}$ is a subgraph of $T^{\prime}$, this
contradicts the minimality of $T^{\prime}$. Hence $T^{\prime}$ does not contain a degree 2 vertex. Therefore, $T^{\prime}$, and thus $T$ contains a jewel.

Below, we obtain a class of trees which attain the lower bound. For a lobster, a limb of the form $K_{1,2}$ is called a pure limb. A lobster with only pure limbs and no whiskers is called a pure lobster.
Corollary 4.3. If $G$ is a pure lobster with spine length $\ell$ and $t$ pure limbs attached to the spine, then $\operatorname{reg}\left(S / J_{G}\right)=\ell+t$.
Proof. Since in a pure lobster, only vertices that have degree 3 or more are in the spine, it can not contain the Jewel graph as a subgraph. Therefore by Theorem 4.1 and Theorem 4.2, we have $\operatorname{reg}\left(S / J_{G}\right)=\operatorname{iv}(G)+1=\ell+t$.

In Theorem 3.1, it was shown that the regularity of the graph obtained by gluing two graphs at a free vertex is sum of the regularities of these two graphs. Naturally, one tends to ask what happens to the regularity if we glue more graphs at a free vertex. We partially answer this question in the next theorem. Let $\mathcal{G}(m, n, w)$ be the family of graphs obtained by identifying a free vertex each of $K_{1, r_{1}}, \ldots, K_{1, r_{m}}$, where $r_{i} \geq 3, n$ cliques on at least


$$
G \in \mathcal{G}(2,2,1)
$$ three vertices and $w$ whiskers.

Theorem 4.4. If $G \in \mathcal{G}(m, n, w), n \geq 2$, then $\operatorname{reg}\left(S / J_{G}\right)=n+2 m$.
Proof. We prove by induction on $m$. Let $m=0$. If $w=0$, then the result follows from Kiani-Madani. Suppose $w \geq 1$. Let $v$ denote the vertex which is common to all the cliques and whiskers. Let $G^{\prime}$ be the clique on $V(G), G^{\prime \prime}$ be the graph induced on $V(G) \backslash\{v\}$ and $H$ be the graph $G^{\prime} \backslash\{v\}$. Then $\operatorname{reg}\left(S / J_{G^{\prime}}\right)=\operatorname{reg}\left(S / J_{H}\right)=1$. Since $G^{\prime \prime}$ is a collection of $n$ disjoint cliques and $w$ isolated vertices, $\operatorname{reg}\left(S / J_{G^{\prime \prime}}\right)=n$. Therefore, the assertion follows from the exact sequence:

$$
\begin{equation*}
0 \rightarrow S / J_{G} \rightarrow S / J_{G^{\prime}} \oplus S / J_{G^{\prime \prime}} \rightarrow S / J_{H} \rightarrow 0 \tag{2}
\end{equation*}
$$

Now assume that $m \geq 1$. Let $\{u\}$ be a leaf vertex in $G$ and $\{u, v\} \in E(G)$. Let $G^{\prime}$ be the graph obtained by adding necessary edges to $G$ so that $N[v]$ is a clique. Let $G^{\prime \prime}$ be the induced subgraph of $G$ on $V(G) \backslash\{v\}$. Let $H$ be the induced subgraph of $G^{\prime}$ on $V\left(G^{\prime}\right) \backslash\{v\}$. Therefore, we have the exact sequence (21). Note that $G^{\prime}, H \in \mathcal{G}(m-1, n+1, w)$ and $G^{\prime \prime} \in \mathcal{G}(m-1, n, w)$. Therefore, by induction hypothesis $\operatorname{reg}\left(S / J_{G^{\prime}}\right)=\operatorname{reg}\left(S / J_{H}\right)=$ $n+2 m-1$ and $\operatorname{reg}\left(S / J_{G^{\prime \prime}}\right)=n+2 m-2$. Therefore, from the short exact sequence, we get $\operatorname{reg}\left(S / J_{G}\right) \leq n+2 m$. Note that $G$ contains $n$ vertex disjoint edges and $m$ vertex disjoint paths length 2 as an induced subgraph. Therefore, $\operatorname{reg}\left(S / J_{G}\right) \geq n+2 m$.

We now consider another subclass of block graphs and obtain an improved upper bound on the regularity of binomial edge ideals of those graphs.

Theorem 4.5. Let $G$ be the union $P \cup C_{1} \cup \cdots \cup C_{r} \cup L_{1} \cup \cdots \cup L_{t} \cup e_{1} \cup \cdots \cup e_{w}$ where $P$ is a longest induced path on the vertices $\left\{v_{0}, \ldots, v_{\ell}\right\}, C_{1}, \ldots, C_{r}$ are maximal cliques on at least three vertices, $L_{1}, \ldots, L_{t}$ be limbs and $e_{1}, \ldots, e_{w}$ are whiskers such that $e_{i} \cap\left\{v_{0}, v_{\ell}\right\}=\emptyset$ and
(1) For all $A, B \in\left\{C_{1}, \ldots, C_{r}, L_{1}, \ldots, L_{t}, e_{1}, \ldots, e_{w}\right\}$ with $A \neq B$,
(a) $A \cap B \subset P$ and $|A \cap B| \leq 1$;
(b) $|A \cap P|=1$.

Then $\operatorname{reg}\left(S / J_{G}\right) \leq \ell+2 t+r$.
Proof. Without loss of generality, we assume that there are no degree 2 vertices in $\left\{v_{0}, \ldots, v_{\ell}\right\}$. Further, we may assume that there is a clique, say $C_{i}$, such that $C_{i} \cap P=\left\{v_{\ell}\right\}$. If not, then $v_{\ell}$ is a leaf in $G$. Attach a clique $C^{\prime}$ to $\left\{v_{\ell}\right\}$, to get a graph $G_{1}$ having $\operatorname{reg}\left(S / J_{G_{1}}\right)=\operatorname{reg}\left(S / J_{G}\right)+1$ (by gluing theorem).

We prove the assertion by induction on $t$. Let $t=0$. We argue this case by induction on $\ell$. Suppose $\ell=0$. Since $e_{i} \cap\left\{v_{0}, v_{\ell}\right\}=\emptyset, w=0$. Hence $G \in \mathcal{G}(0, r, 0)$. Hence the result follows from Theorem 4.4.

Let $\ell=1$. Suppose $v_{0} \in C_{1}$. Let $G^{\prime}, G^{\prime \prime}$ and $H$ be the graphs obtained by applying EHH-process on $G$ with respect to $v_{0}$. Then $G^{\prime \prime}$ is a block graph with exactly $r$-cliques, $G^{\prime}$ and $H$ are block graphs with at most $r$-cliques. Therefore, it follows from the exact sequence (1) and [9, Theorem 3.5] that

$$
\operatorname{reg}\left(S / J_{G}\right) \leq \max \left\{\operatorname{reg}\left(S / J_{G^{\prime}}\right), \operatorname{reg}\left(S / J_{G^{\prime \prime}}\right), \operatorname{reg}\left(S / J_{H}\right)+1\right\} \leq r+1
$$

Now, suppose $\ell \geq 2$. Without loss of generality, assume that $C_{1} \cap\left\{v_{\ell-1}, v_{\ell}\right\} \neq \emptyset$. Let $G^{\prime}, G^{\prime \prime}$ and $H$ be the graphs obtained by applying EHH-process on $G$ with respect to $v_{\ell-1}$. Then $G^{\prime}$ is the union of path $P^{\prime}$ of length at most $\ell-1$ containing $\left\{v_{0}, \ldots, v_{\ell-2}\right\}$ and cliques, $\left\{C_{1}^{\prime}, C_{2}, \ldots, C_{r}\right\}$ and a subset of whiskers $\left\{e_{1}, \ldots, e_{w}\right\}$. Hence by induction, $\operatorname{reg}\left(S / J_{G^{\prime}}\right) \leq \ell-1+r$. Similarly $\operatorname{reg}\left(S / J_{H}\right) \leq \ell-1+r$ and $\operatorname{reg}\left(S / J_{G^{\prime \prime}}\right) \leq \ell-1+r$. Therefore, it follows from the exact sequence (11) that $\operatorname{reg}\left(S / J_{G}\right) \leq \ell+r$.

Let us assume that $t \geq 1$. Since there are no degree 2 vertices in $G$, each $\operatorname{limb} L_{i}=K_{1, \mu_{i}}$ for some $\mu_{i} \geq 3$. Consider the limb $L_{t}$. Suppose $L_{t} \cap P=\{v\}$ and $N_{L_{t}}(v)=\{u\}$. Let $G^{\prime}, G^{\prime \prime}$ and $H$ be the graphs obtained by applying EHH-process on $G$ with respect to $u$. Then $G^{\prime}$ and $H$ both have spines of length $\ell, r+1$ cliques, $t-1$ limbs and $w$ whiskers. Also, $G^{\prime \prime}$ has spine of length $\ell, r$ cliques, $t-1$ limbs and $w$ whiskers. Therefore, by induction, $\operatorname{reg}\left(S / J_{G^{\prime}}\right), \operatorname{reg}\left(S / J_{H}\right) \leq \ell+(r+1)+2(t-1)$, and $\operatorname{reg}\left(S / J_{G^{\prime \prime}}\right) \leq \ell+r+2(t-1)$. Hence, from the exact sequence (1), it follows that $\operatorname{reg}\left(S / J_{G}\right) \leq \ell+(r+1)+2(t-1)+1=\ell+r+2 t$ as required.

Note that the graphs $G$ considered in Theorem 4.5 are block graphs, and hence by Theorem 3.5 of [9], one has $\operatorname{reg}\left(S / J_{G}\right) \leq c(G)$, where $c(G)$ is the number of cliques in $G$. In the case where the $L_{i}=K_{1, r_{i}}$ for $r_{i} \geq 3$ and $w>0$, the bound given above is much smaller to the Madani-Kiani bound.

As a consequence of the above theorem, we generalize a result of Chaudhry et al. to obtain an upper bound on the regularity of lobster graphs.
Corollary 4.6. If $G$ is a lobster with spine $P$ of length $\ell$ and $t$ limbs $P$, then $\operatorname{reg}\left(S / J_{G}\right) \leq$ $\ell+2 t$.

Proof. Take $r=0$ in Theorem 4.5.

Example 4.7. This is an example of a lobster which attains the upper bound given in Theorem 4.6.


This graph G has many different longest induced paths. Fixing any one of them, one can see that $G$ has spine length $\ell=4, t=4$ limbs attached to the spine and 2 whiskers. It can be shown that

$$
\operatorname{reg}\left(S / J_{G}\right)=12=\ell+2 t
$$

Figure 3. G
Corollary 4.8. Let $G$ be a lobster with spine $P$ of length $\ell, t$ limbs and $r$ whiskers. Then $\ell+t \leq \operatorname{reg}\left(S / J_{G}\right) \leq \ell+2 t$.

Proof. The upper bound is proved in Corollary 4.6. To prove the lower bound, note that $G$ has a subgraph $G^{\prime}$ with spine $P, t$ pure limbs and without any whiskers as an induced subgraph. By Corollary 4.3, $\operatorname{reg}\left(S / J_{G^{\prime}}\right)=\ell+t$ as required.

From Theorem 4.2, it is clear that the presence of Jewel graph as a subgraph plays crucial role in determining the regularity of a tree. It can be seen that $\operatorname{reg}\left(S / J_{T}\right) \geq$ $\operatorname{iv}(T)+j$, where $T$ contains $j$ vertex disjoint copies of the Jewel graph. We believe that understanding the regularity behaviour of collection of Jewels that share vertices and/or edges can lead to a precise estimation of regularity of trees.

Recently, Herzog and Rinaldo has generalized Theorem 4.1 to certain block graphs.
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