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Quantum Bernstein fractal functions

N. Vijender[1] | **A.K.B. Chand**[2] | **M. A. Navascués**[3] | **M. V. Sebastián**[4]

[1]Department of Basic Sciences and Engineering, Indian Institute of Information Technology Nagpur, India

[2]Department of Mathematics, Indian Institute of Technology Madras, India

[3]Departamento de Matemática Aplicada, Universidad de Zaragoza, Spain

[4]Centro Universitario de la Defensa, Academia General Militar, Spain

Correspondence
M.V. Sebastián, Centro Universitario de la Defensa, Academia General Militar, Ctra de Huesca s/n, 50090 Zaragoza, Spain.
Email: msebasti@unizar.es

In this article, taking the quantum Bernstein functions as base functions, we have proposed the class of quantum Bernstein fractal functions. When $f \in \mathcal{C}(I)$, the base function is taken as the classical q-Bernstein polynomials, we propose the class of quantum fractal functions through a multivalued quantum fractal operator. When $f \in \mathcal{L}^p(I), 1 \leq p \leq \infty$, the base function is assumed as q-Kantorovich-Bernstein polynomial to construct the sequence of (q, α)-Kantorovich-Bernstein fractal functions that converges uniformly to f. Some approximation properties of these new class of quantum fractal interpolants have been studied.

KEYWORDS

q-Bernstein polynomials, constrained approximation, fractal functions, fractals, interpolation

MOS SUBJECT CLASSIFICATION

28A80; 26C15; 26A48; 26A51; 65D05

1 | INTRODUCTION

Quantum calculus (in short q-calculus) is in the homework of the classical infinitesimal calculus without the notion of limit. It works as a bridge between mathematics and physics for the last five decades. One can find its application in different mathematical areas such as number theory, combinatorics, orthogonal polynomials, basic hypergeometric functions apart from quantum theory, mechanics, and theory of relativity in physics. In 1912, using polynomials, Bernstein gave an alternative proof of the Weierstrass theorem: Every continuous function on $[a,b]$ can be uniformly approximated by a sequence of polynomial functions. Since Bernstein polynomials play an important role in approximation theory, many researchers have studied this polynomial and its different generalizations, see for instance References 1-3. Lupas[4] first introduced the q-analogue of Bernstein polynomials that brought into the existence of a new research area called\ q-approximation theory. Numerous authors[5-12] have investigated and proposed the q-extension of various results of classical approximation theory.

However, classical approximation theory and q-approximation theory deal with the approximation of functions using smooth functions or infinitely differentiable functions. However, the classical smooth functions may not provide good representatives of irregular functions, for instance, Weierstrass function, and real-world sampled signals such as financial series, seismic data, speech signals, bioelectric recordings, and so on. Fractal functions provide a constructive approximation theory for nondifferentiable functions. Fractal functions concern mainly at data/function which present details at different scales or some degree of self-similarity.

By exploiting the theory of iterated function system (IFS),[13] Barnsley[14] introduced the concept of fractal interpolation function (FIF) to provide a mathematical representation of a data set that is generated from irregularity and/or self-affine structure. The calculus of fractal functions was investigated in References 15,16 and this research provided a methodology for the construction of C^r-fractal splines. Very recently, shape-preserving fractal interpolation was studied in

References 17-20. In these articles various types of fractal splines that preserve the fundamental shapes of the interpolation data were developed. Shape-preserving fractal surfaces and their convergence and stability aspects were investigated in References 21-23. Furthermore, Barnsley[14] has extended the idea of fractal interpolation to approximate a continuous function f defined on a real compact interval I, and this led to the concept of fractal approximation or α-fractal function f^α of f.[24-26] In general, (i) α-fractal functions are nondifferentiable; (ii) the graph of f^α is a union of transformed copies of itself; (iii) fractal dimension of graph of α-fractal function is noninteger. Due to these fractal characteristics, f^α may be treated as the fractal approximant of f. In this way, every continuous function can be approximated by means of fractal functions. Furthermore, shape-preserving fractal approximation was investigated in Reference 27. Akhtar et al[28] calculated the box dimension of the graph of α-fractal functions by assuming suitable conditions on the original function f and base function.

Navascués et al[24-26,29-31] studied various properties of the α-fractal function f^α of f including approximation properties, among various desirable properties of a good approximant. Navascués et al[24-26,29-31] proved that the α-fractal function f^α of f converges toward f provided the magnitude of the scaling factors of f^α goes to zero. In this article, using the theory of fractal functions and classical q-approximation, for a given function $f \in \mathcal{C}(I)$, we propose a sequence $\{f_n^{(q,\alpha)}\}_{n=1}^{\infty}$ of quantum fractal functions that converges to f even if the magnitude/norm of the corresponding scaling factors/functions does not go to zero. In the construction of the sequence $\{f_n^{(q,\alpha)}\}_{n=1}^{\infty}$ of quantum fractal functions, we use the sequence $\{B_{n,q}(f,\cdot)\}_{n=1}^{\infty}$ of q-Bernstein polynomials of f as base functions. Consequently, the convergence of the sequence $\{f_n^{(q,\alpha)}\}_{n=1}^{\infty}$ of quantum fractal functions toward the function f follows from the convergence of the q-Bernstein polynomials toward f. The shape of the quantum fractal functions depends on the choice of $q \in (0,1)$ and the scaling functions. When $q \to 1$, the q-Bernstein polynomial coincides with the classical Bernstein polynomial, and in this case we call quantum fractal functions simply α-fractal functions. Furthermore, the convergence of these α-fractal functions toward f follows from the convergence of the q-Bernstein polynomials of f toward f. The procedure of getting a sequence $\{f_n^{(q,\alpha)}\}_{n=1}^{\infty}$ of quantum fractal functions that converges uniformly to $f \in \mathcal{C}(I)$ determines an operator, termed the multivalued quantum fractal operator: $\mathcal{F}^{(q,\alpha)} : \mathcal{C}(I) \rightrightarrows \mathcal{C}(I), f \to \{f_n^{(q,\alpha)}\}_{n=1}^{\infty}$. We study some basic properties of $\mathcal{F}^{(q,\alpha)}$.

Navascués and Chand[29] extended the notion of α-fractal function to $\mathcal{L}^p$-spaces and derived some approximation results under the assumption that the norm of the scaling functions tends to zero. In this article, we develop (q,α)-Kantorovich-Bernstein fractal functions in $\mathcal{L}^p$-spaces without any condition on the scaling functions for convergence. Furthermore, we study the approximation properties of (q,α)-Kantorovich-Bernstein fractal functions and quantum fractal versions of Müntz theorems in $\mathcal{L}^p$-spaces.

2 | BACKGROUND AND PRELIMINARIES

In this section we endeavor to expose the reader to the requisite preliminaries on fractal interpolation functions and its generalization through α-fractal functions.

2.1 | Fractal interpolation

Let $\mathbb{N}_k$ denote the first k natural numbers, $I = [x_1, x_N]$ be a closed and bounded interval of $\mathbb{R}$, and $\mathcal{C}(I)$ be the Banach space of all real-valued continuous functions on I equipped with the supremum norm. Consider the interpolation data $\{(x_i, y_i) : i \in \mathbb{N}_N\}$ with strictly abscissae and $N > 2$. Let $L_i, i \in \mathbb{N}_{N-1}$, be a set of homeomorphic mappings from I to $I_i = [x_i, x_{i+1}]$ satisfying

$$L_i(x_1) = x_i, \quad L_i(x_N) = x_{i+1}. \tag{1}$$

Let F_i be a function from $I \times K$ to K (K is suitable compact subset of $\mathbb{R}$), which is continuous in the x-direction and contractive in the y-direction (with contractive factor $|\alpha_i| \le \kappa < 1$) such that

$$F_i(x_1, y_1) = y_i, \quad F_i(x_N, y_N) = y_{i+1}, \quad i \in \mathbb{N}_{N-1}. \tag{2}$$

Let us consider $\mathcal{G} = \{g \in \mathcal{C}(I) | g(x_1) = y_1 \text{ and } g(x_N) = y_N\}$. We define a metric on $\mathcal{G}$ by $\rho(h,g) = \max\{|h(x) - g(x)| : x \in I\}$ for $h, g \in \mathcal{G}$. Then $(\mathcal{G}, \rho)$ is a complete metric space. Define the Read-Bajraktarević operator T on $(\mathcal{G}, \rho)$ by

$$Tg(x) = F_i(L_i^{-1}(x), g \circ L_i^{-1}(x)), \quad x \in I_i. \tag{3}$$

Using the properties of L_i and (1) and (2), Tg is continuous on the interval $I_i; i \in \mathbb{N}_{N-1}$, and at each of the points $x_2, \dots, x_{N-1}$. Also,

$$\rho(Tg, Th) \leq |\alpha|_\infty \rho(g, h),$$

where $|\alpha|_\infty = \max\{|\alpha_i| : i \in \mathbb{N}_{N-1}\} < 1$. Hence, T is a contraction map on the complete metric space $(\mathcal{G}, \rho)$. Therefore, by the Banach fixed point theorem, T possesses a unique fixed point (say) f^* on $\mathcal{G}$, that is, $(Tf^*)(x) = f^*(x)$ for all $x \in I$. According to (3), the function f^* satisfies the functional equation: $f^*(x) = F_i(L_i^{-1}(x), f^* \circ L_i^{-1}(x))$, $x \in I_i$. Furthermore, using (1) and (2), it is easy to verify that $f^*(x_i) = y_i$, $i \in \mathbb{N}_N$. Defining a mapping $w_i : I \times K \to I_i \times K$ as $w_i(x, y) = (L_i(x), F_i(x, y))$, $(x, y) \in I \times K$, $i \in \mathbb{N}_{N-1}$, the graph $G(f^*)$ of f^* satisfies:

$$G(f^*) = \bigcup_{i \in \mathbb{N}_{N-1}} w_i(G(f^*)), \tag{4}$$

and hence f^* is called fractal interpolation function (FIF) corresponding to the IFS $\mathcal{I} = \{I \times K, w_i(x, y) = (L_i(x), F_i(x, y)), i \in \mathbb{N}_{N-1}\}$.

Barnsley and Navascués[14,24,25] observed that the concept of FIFs can be used to define a class of fractal functions associated with a given function $f \in C(I)$.

For a given $f \in C(I)$, consider a partition $\Delta = \{x_1, x_2, \dots, x_N\}$ of $[x_1, x_N]$ satisfying $x_1 < x_2 < \dots < x_N$, a continuous function $b : I \to \mathbb{R}$ that fulfills the conditions $b(x_1) = f(x_1)$, $b(x_N) = f(x_N)$ and $b \neq f$, and $N - 1$ real numbers α_i, $i \in \mathbb{N}_{N-1}$ satisfying $|\alpha_i| < 1$. Define an IFS through the maps

$$L_i(x) = a_i x + b_i, \quad F_i(x, y) = \alpha_i y + f(L_i(x)) - \alpha_i b(x), \quad i \in \mathbb{N}_{N-1}.$$

The corresponding FIF denoted by $f^\alpha_{\Delta,b} = f^\alpha$ is referred to as α-fractal function for f (fractal approximation of f) with respect to a scaling vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{N-1})$, base function b, and partition Δ. Here the set of data points is $\{(x_i, f(x_i)) : i \in \mathbb{N}_N\}$. The function f^α is the fixed point of the Read-Bajraktarević (RB) operator $T : \mathcal{C}_f(I) \to \mathcal{C}_f(I)$ defined by

$$(Tg)x = \alpha_i g(L_i^{-1}(x)) + f(x) - \alpha_i b(L_i^{-1}(x)), \quad x \in I_i, \quad i \in \mathbb{N}_{N-1},$$

where $\mathcal{C}_f(I) = \{g \in \mathcal{C}(I) : g(x_1) = f(x_1),\ g(x_N) = f(x_N)\}$. Consequently, the α-fractal function f^α corresponding to f satisfies the self-referential equation

$$f^\alpha(x) = \alpha_i f^\alpha(L_i^{-1}(x)) + f(x) - \alpha_i b(L_i^{-1}(x)), \quad x \in I_i, \quad i \in \mathbb{N}_{N-1}. \tag{5}$$

The fractal dimension (box dimension or Hausdorff dimension) of f^α depends on the choice of the scaling vector α. For instance, Akhtar et al[28] calculated box dimension of graph of α-fractal functions by assuming suitable conditions on the original function f and base function b. The following proposition provides the details of it.

Proposition 1. *Let $f \in \mathcal{C}(I)$ and $b : I \to \mathbb{R}$ be Lipschitz functions with $b(x_1) = f(x_1)$, $b(x_N) = f(x_N)$. Let $\Delta = \{x_1, x_2, \dots, x_N\}$ be a partition of I satisfying $x_1 < x_2 < \dots < x_N$. and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{N-1})$. If the data points $(x_i, f(x_i)), i \in \mathbb{N}_N$ are not collinear, then graph G of the α-fractal function f^α has the box dimension*

$$dim_B(G) = \begin{cases} D & \text{if } \sum_{i=1}^{N-1} |\alpha_i| > 1, \\ 1 & \text{otherwise,} \end{cases}$$

where D is solution of $\sum_{i=1}^{N-1} |\alpha_i| a_i^{D-1} = 1$.

To obtain fractal functions with more flexibility, iterated function system wherein scaling factors are replaced by scaling functions received attention in the recent literature[32] on fractal functions. That is, one may consider the IFS with maps

$$L_i(x) = a_i x + b_i, \quad F_i(x, y) = \alpha_i(x) y + f(L_i(x)) - \alpha_i b(x), \quad i \in \mathbb{N}_{N-1},$$

where $\alpha_i, i \in \mathbb{N}_{N-1}$ are continuous functions on I satisfying $\max\{\|\alpha_i\|_\infty : \iota \in \mathbb{N}_{N-1}\} < 1$. The corresponding α-fractal function is the fixed point of the RB-operator

$$(Tg)x = \alpha_i(L_i^{-1}(x))g(L_i^{-1}(x)) + f(x) - \alpha_i(L_i^{-1}(x))b(L_i^{-1}(x)), \quad x \in I_i, \quad i \in \mathbb{N}_{N-1}. \tag{6}$$

Consequently, the α-fractal function f^α corresponding to f satisfies the self-referential equation

$$f^\alpha(x) = \alpha_i(L_i^{-1}(x))f^\alpha(L_i^{-1}(x)) + f(x) - \alpha_i(L_i^{-1}(x))b(L_i^{-1}(x)), \quad x \in I_i, \quad i \in \mathbb{N}_{N-1}. \tag{7}$$

3 | QUANTUM FRACTAL APPROXIMATION

From (7), we get the following inequality:

$$\|f^\alpha - f\|_\infty \leq \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty}\|f - b\|_\infty, \tag{8}$$

where $\|\alpha\|_\infty = \max\{\|\alpha_i\|_\infty : i \in \mathbb{N}_{N-1}\}$. For a fixed base function b, the α-fractal function f^α converges uniformly to $f \in C(I)$ if $\|\alpha\|_\infty \to 0$. To get the convergence of the α-fractal function f^α toward f without altering the scaling functions, we choose the base function b as q-Bernstein polynomial $B_{n,q}(f,x)$ of f, that is, $b = B_{n,q}(f,x)$ (see for instance Reference 33),

$$B_{n,q}(f,x) = \frac{1}{(x_N - x_1)^n}\sum_{k=0}^{n}\binom{n}{k}_q (x - x_1)^k f\left(x_1 + (x_N - x_1)\frac{[k]_q}{[n]_q}\right)\prod_{s=0}^{n-k-1}(x_N - x_1 - q^s x), \quad x \in I, \tag{9}$$

where $q \in (0,1), n \in \mathbb{N}$, $[k]_q = \frac{1-q^k}{1-q}$,

$$[k]_q! = \begin{cases} [k]_q[k-1]_q[k-2]_q \ldots [2]_q[1]_q, & \text{if } k \neq 0,\\ 1, & \text{if } k = 0,\end{cases}$$

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad f \in C(I), \quad B_{n,q}(f,x_1) = f(x_1), \quad B_{n,q}(f,x_N) = f(x_N).$$

When $q \to 1$, $B_{n,q}(f,x)$ coincides with the classical nth Bernstein polynomial. If we take the base function as $b = B_{n,q}(f,x)$ in (9), then the corresponding fractal function $\mathcal{F}^{q,\alpha}_{\Delta,B_n}(f) = f_n^{(q,\alpha)}$ is called a quantum Bernstein fractal function associated with $f \in C(I)$, and

$$f_n^{(q,\alpha)}(x) = f(x) + \alpha_i(L_i^{-1}(x))[f_n^{(q,\alpha)}(L_i^{-1}(x)) - B_{n,q}(f, L_i^{-1}(x))], \quad x \in I_i, \quad i \in \mathbb{N}_{N-1}, \quad n \in \mathbb{N}. \tag{10}$$

Therefore, from (10), it is easy to notice that shape and properties of the quantum fractal function $f_n^{(q,\alpha)}$ depend on the choice of $q \in (0,1)$ apart from the choice of scaling functions. Note that the quantum fractal function $f_n^{(q,\alpha)}$, $n \in \mathbb{N}$, of $f \in C(I)$ is obtained via the IFS defined by

$$\mathcal{I}_n = \{I \times \mathbb{R}, (L_i(x), F_{n,i}(x,y)) : i \in \mathbb{N}_{N-1}\}, \quad n \in \mathbb{N}, \tag{11}$$

where $F_{n,i}(x,y) = f(L_i(x)) - \alpha_i(x)(y - B_{n,q}(f,x))$.

Theorem 1. *Let $f \in C(I)$. There exists a sequence of quantum Bernstein fractal functions $\{f_n^{(q,\alpha)}(x)\}_{n=1}^\infty$ that converges uniformly to f on I. Furthermore, $f_n^{(q,\alpha)}$, $n \in \mathbb{N}$, satisfies the following inequalities:*

$$\frac{1 - \|\alpha\|_\infty}{1 + \|\alpha\|_\infty}\|f\|_\infty \leq \|f_n^{(q,\alpha)}\| \leq \frac{1 + \|\alpha\|_\infty}{1 - \|\alpha\|_\infty}\|f\|_\infty. \tag{12}$$

Proof. Let $f_n^{(q,\alpha)}$, $n \in \mathbb{N}$, be the quantum fractal function corresponding to f. Then, from (10), it is easy to deduce that

$$\begin{aligned}\|f_n^{(q,\alpha)} - f\|_\infty &\le \|\alpha\|_\infty \|f_n^{(q,\alpha)} - B_{n,q}(f,.)\|_\infty, \\ &\le \|\alpha\|_\infty [\|f_n^{(q,\alpha)} - f\|_\infty + \|f - B_{n,q}(f,.)\|_\infty].\end{aligned}$$

Hence we obtain

$$\|f_n^{(q,\alpha)} - f\|_\infty \le \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|f - B_{n,q}(f,.)\|_\infty. \tag{13}$$

From Reference 33, we have

$$\|B_{n,q}(f,.) - f\|_\infty \to 0 \text{ as } n \to \infty. \tag{14}$$

Using (14) in (13), we conclude that the sequence $\{f_{n,q}(x)\}_{n=1}^{\infty}$ of quantum fractal functions converges uniformly to f. Again from References 34, we have

$$\|B_{n,q}(.,.)\|_\infty = 1, \quad q \in (0,1]. \tag{15}$$

We can rewrite (13) as

$$\|f_n^{(q,\alpha)}\|_\infty - \|f\|_\infty \le \|f_n^{(q,\alpha)} - f\|_\infty \le \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\{\|f\|_\infty + \|B_{n,q}(f,.)\|_\infty\}. \tag{16}$$

Using (15) in (16), we get the right-side inequality of (12). Next, from (10), we obtain

$$|f_n^{(q,\alpha)}(x) - f(x)| \le \|\alpha_i\|_\infty \{\|f_n^{(q,\alpha)}\|_\infty + \|B_{n,q}(f,.)\|_\infty\}, \quad x \in I_i, \quad i \in \mathbb{N}_{N-1}, n \in \mathbb{N},$$

which implies that

$$\|f\|_\infty - \|f_n^{(q,\alpha)}\|_\infty \le \|f_n^{(q,\alpha)} - f\|_\infty \le \|\alpha\|_\infty \{\|f_n^{(q,\alpha)}\|_\infty + \|B_{n,q}(f,.)\|_\infty\}.$$

Using (15) in the above inequality, we get the left-side inequality of (12). ∎

Proposition 2. *If we consider $\mathcal{L}^p$-norm $\|f\|_{\mathcal{L}^p} = \left(\int_I |f(t)|^p dt\right)^{1/p}$, $1 < p < \infty$, for $f \in C(I)$, the following inequality holds.*

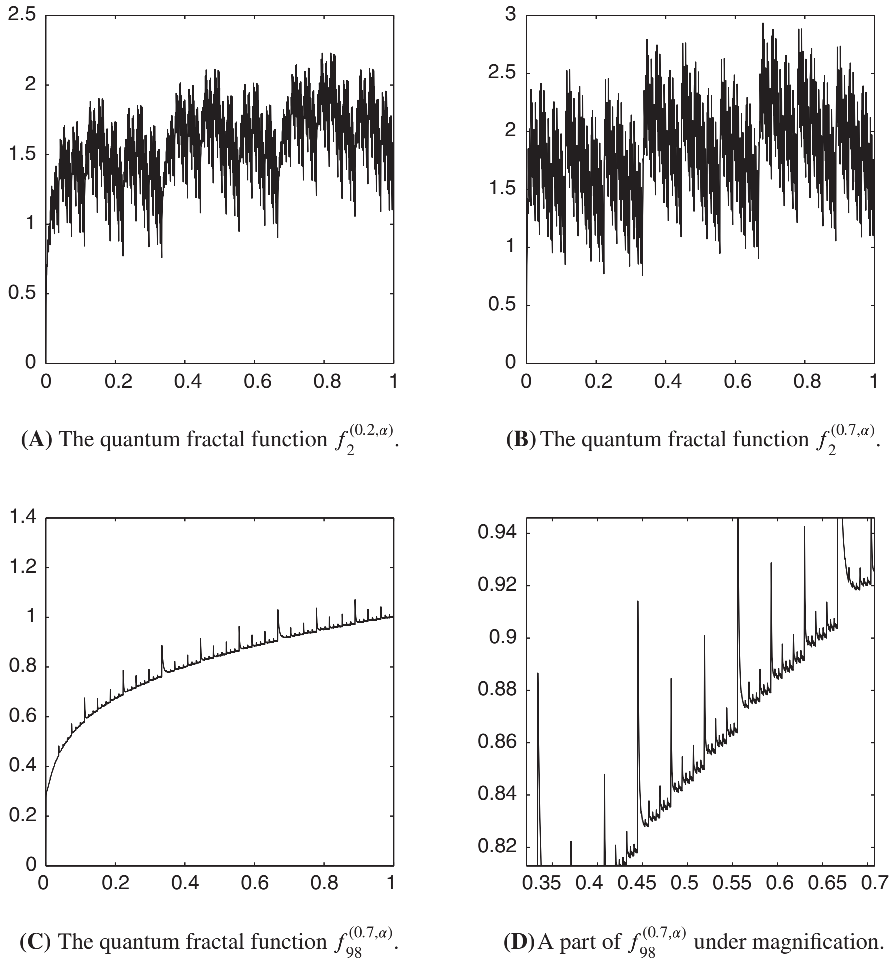
$$\|f_n^{(q,\alpha)} - f\|_{\mathcal{L}^p} \le \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|f - B_{n,q}(f)\|_{\mathcal{L}^p}. \tag{17}$$

Proof. From (10), we have

$$\begin{aligned}\|f_n^{(q,\alpha)} - f\|_{\mathcal{L}^p}^p &= \int_I |(f_n^{(q,\alpha)} - f)(x)|^p dx \\ &= \sum_{i=1}^{N-1}\int_{x_i}^{x_{i+1}} |\alpha_i(L_i^{-1}(x)|^p |(f_n^{(q,\alpha)} - B_{n,q}(f))\circ L_i^{-1}(x)|^p dx \\ &= \sum_{i=1}^{N-1}\int_I a_i |\alpha_i(\tilde{x})|^p |(f_n^{(q,\alpha)} - B_{n,q}(f))(\tilde{x})|^p d\tilde{x} \\ &\le \sum_{i=1}^{N-1} a_i \|\alpha\|_\infty^p \int_I |(f_n^{(q,\alpha)} - B_{n,q}f)(x)|^p dx \\ &= \|\alpha\|_\infty^p \|f_n^{(q,\alpha)} - B_{n,q}(f)\|_{\mathcal{L}^p}^p.\end{aligned}$$

In the above computation, we have used the change of variable $\tilde{x} = L_i^{-1}(x)$ at the third step and $\sum_{N-1}^{i=1} a_i = 1$ at the final step. From the above estimation, we have

$$\begin{aligned}\|f_n^{(q,\alpha)} - f\|_{\mathcal{L}^p} &\le \|\alpha\|_\infty \|f_n^{(q,\alpha)} - B_{n,q}(f)\|_{\mathcal{L}^p} \\ &\le \|\alpha\|_\infty (\|f_n^{(q,\alpha)} - f\|_{\mathcal{L}^p} + \|f - B_{n,q}(f)\|_{\mathcal{L}^p}).\end{aligned}$$



(A) The quantum fractal function $f_2^{(0.2,\alpha)}$.

(B) The quantum fractal function $f_2^{(0.7,\alpha)}$.

(C) The quantum fractal function $f_{98}^{(0.7,\alpha)}$.

(D) A part of $f_{98}^{(0.7,\alpha)}$ under magnification.

FIGURE 1 The quantum fractal approximants of $x^{1/4}, x \in [0,1]$

Further simplification of the above inequality gives the desired estimation in (17). ■

Examples. Now, we want to see some examples of q-fractal functions for a given function $f(x) = x^{1/4}, x \in [0,1]$. The quantum fractal functions in Figure 1A-C are generated with respect to the partition $\Delta = \{0, 0.25, 0.5, 1\}$ of [0,1]. The quantum fractal functions $f_2^{(0.2,\alpha)}, f_2^{(0.7,\alpha)}$, and $f_{98}^{(0.7,\alpha)}$ are generated at the sixth iteration, respectively, in Figure 1A-C with the choice of the scaling functions $\alpha_i(x) = \frac{1}{1+e^{-10x}}, x \in [0,1], i \in \mathbb{N}_3$. By comparing the quantum fractal functions $f_2^{(0.2,\alpha)}$ and $f_2^{(0.7,\alpha)}$, one can observe the effects of q in the shape of the quantum fractal function. According to Theorem 1, the quantum fractal function $f_{31}^{(0.7,\alpha)}$ provides a better approximation for $x^{1/4}, x \in [0,1]$ than that obtained by $f_2^{(0.7,\alpha)}$. By observing Figure 1B,C, one can ask why the fractal functions $f_2^{(0.7,\alpha)}$ and $f_{98}^{(0.7,\alpha)}$ do not have the same sort of irregularity even if their scaling functions are same. This is due to the following reason: The fractal function $f_2^{(0.7,\alpha)}$ exhibits irregularity on all scales, whereas the fractal function $f_{31}^{(0.7,\alpha)}$ exhibits irregularity on small scales. Furthermore, small scales of irregularity of the fractal function $f_{98}^{(0.7,\alpha)}$ can be observed from Figure 1D, which is a part of $f_{98}^{(0.7,\alpha)}$ under magnification.

4 | MULTIVALUED QUANTUM FRACTAL OPERATOR

The definition of quantum α-fractal function $\mathcal{F}_{\Delta,B_n}^{(q,\alpha)}(f) = f_{\Delta,B_n}^{(q,\alpha)} = f_n^{(q,\alpha)}$ corresponding to each $f \in C(I)$ yields a multivalued quantum fractal operator $\mathcal{F}^{(q,\alpha)} : C(I) \rightrightarrows C(I)$ defined by

$$\mathcal{F}^{(q,\alpha)}(f) = \{\mathcal{F}_n^{(q,\alpha)}(f)\}_{n=1}^{\infty} = \{f_n^{(q,\alpha)}\}_{n=1}^{\infty}.$$

Let us record some definitions which are needed for our further investigations.

Definition 1 (35). Let X and Y be two real normed linear spaces over $\mathbb{R}$. For a multivalued map $T : X \to Y$, the domain of T is defined by $Dom(T) = \{x \in X : T(x) \neq \emptyset\}$. Then $T : X \rightrightarrows Y$ is

- convex if for all $x_1, x_2 \in Dom(T)$ and for all $\lambda \in [0, 1]$,

$$\lambda T(x_1) + (1 - \lambda)T(x_2) \subseteq T(\lambda x_1 + (1 - \lambda)x_2).$$

- process if for all $x \in Dom(T)$ and for all $\lambda > 0$,

$$T(\lambda x) = \lambda T(x) \quad \text{and} \quad 0 \in T(0).$$

- linear if for all $x_1, x_2 \in Dom(T)$ and for all $\beta, \gamma \in \mathbb{R}$,

$$\beta T(x_1) + \gamma T(x_2) \subseteq T(\beta x_1 + \gamma x_2).$$

- Lipschitz if there exists a constant $\nu > 0$ such that for all $x_1, x_2 \in Dom(T)$

$$T(x_1) \subseteq T(x_2) + \nu \|x_1 - x_2\| U_Y,$$

  where U_Y is the closed unit ball in Y.

Theorem 2 (36, corollary 1.4). *Let X and Y be real vector spaces and $P_0(Y)$ be the collection of all nonempty subsets of Y. A multivalued map $T : X \to P_0(Y)$ is linear and $T(0) = \{0\}$ if and only if T is single-valued map.*

Theorem 3 (36, corollary 1.4). *Let X and Y be real vector spaces and $P_0(Y)$ be the collection of all nonempty subsets of Y. If a multivalued map $T : X \to P_0(Y)$ is such that $T(x_0)$ is a singleton for some $x_0 \in X$, then $T : X \to P_0(Y)$ is convex if and only if T is single-valued and affine.*

Theorem 4. *The multivalued quantum fractal operator $\mathcal{F}^{(q,\alpha)} : C(I) \rightrightarrows C(I)$ defined by $\mathcal{F}^{(q,\alpha)}(f) = \{f_n^{(q,\alpha)}\}_{n=1}^{\infty}$ is not linear.*

Proof. Clearly $\mathcal{F}^{(q,\alpha)}$ is multivalued. Also, from definition $\mathcal{F}^{(q,\alpha)}(0) = \{0\}$. Hence, by Theorem 2, $\mathcal{F}^{(q,\alpha)}$ is not linear. ■

Remark 1. Note that $B_{n,q} : (\mathcal{L}^p, \|\cdot\|_p) \to (\mathcal{L}^p, \|\cdot\|_p)$ is not bounded on $(\mathcal{L}^p(I), \|\cdot\|_p)$. Thus, we cannot use the standard density argument to extend the continuous quantum Bernstein fractal functions to $(\mathcal{L}^p(I), \|\cdot\|_p)$. Therefore, we will construct $\mathcal{L}^p$-quantum fractal Bernstein fractal functions by using Kantorovich-Bernstein polynomials in the following.

5 | Kantorovich-Bernstein fractal functions in $\mathcal{L}^p$ spaces

In this section, for a given function $f \in \mathcal{L}^p(I),\ 1 \le p \le \infty$, using q-Kantorovich-Bernstein operator $\Phi_{q,n}$[33] as base function, we develop (q, α)-Kantorovich-Bernstein fractal functions in the following:

It is known[33] that for $f \in \mathcal{L}^p(I),\ 1 \le p \le \infty$, $\|f - \Phi_{q,n}(f)\|_p \to 0$ as $n \to \infty$, where

$$\Phi_{q,n}(f; x) = \frac{1}{(x_N - x_1)^n} \sum_{k=0}^{n} \binom{n}{k}_q (x - x_1)^k (x_N - x)^{n-k} [n+1]_q \int_{x_1 + \frac{k(x_N - x_1)}{[n+1]_q}}^{x_1 + \frac{(k+1)(x_N - x_1)}{[n+1]_q}} f(t) d_q t,$$

where $d_q t$ denotes the q-integration.[37]

The proof of the following theorem can be obtained using the arguments similar to those used in Reference 38.

Theorem 5. *Let $f \in \mathcal{L}^p(I),\ 1 \le p \le \infty$. Suppose $\Delta = \{x_1, x_2, \ldots, x_N\}$ be a partition of I satisfying $x_1 < x_2 < \ldots < x_N$, $I_i := [x_i, x_{i+1}),\ i \in \mathbb{N}_{N-2}$, $I_{N-1} = [x_{N-1}, x_N]$. Let $L_i(x) = a_i x + b_i$ satisfy (1). If $\alpha_i \in \mathcal{L}^\infty(I)$ for all $i \in \mathbb{N}_{N-1}$ and $b(x) = \Phi_{q,n}(f; x) \in \mathcal{L}^p(I)$, then the RB-operator given in (6) maps $\mathcal{L}^p(I)$ onto itself. Furthermore, if the scaling function satisfies the* *condition*

$$\begin{cases} \left[\sum_{i\in\mathbb{N}_{N-1}} a_i\|\alpha_i\|_\infty^p\right]^{\frac{1}{p}} < 1 & \text{if } 1\le p<\infty, \\ \|\alpha\|_\infty < 1 & \text{if } p=\infty, \end{cases}$$

then T is a contraction on $\mathcal{L}^p$, and gives a fixed point $f_n^{(q,\alpha)} \in \mathcal{L}^p(I)$ for each $n\in\mathbb{N}$, which satisfies the self-referential equation (4).

From here we will assume that these conditions on the scaling functions are satisfied.

We define a (q,α)-Kantorovich-Bernstein fractal function as the solution of the fixed point equation:

$$f_n^{(q,\alpha)}(x) = f(x) + (f_n^{(q,\alpha)}(L_i^{-1}(x)) - \Phi_{q,n}(f;L_i^{-1}(x)))\alpha_i(L_i^{-1}(x)) \quad \forall x\in I_i, \quad n\in\mathbb{N}, \quad i\in N_{N-1}. \tag{18}$$

Theorem 6. *For $f\in\mathcal{L}^p(I)$ and the scaling functions satisfying the conditions given in Theorem 5, there exists a sequence $\{f_n^{(q,\alpha)}(x)\}_{n=1}^{\infty}$ of (q,α)-Kantorovich-Bernstein fractal functions that converges uniformly to f on I.*

Proof. From (18) for $1\le p<\infty$, we obtain

$$\begin{aligned} \|f_n^{(q,\alpha)} - f\|_p^p &= \int_I |(f_n^{(q,\alpha)} - f)(x)|^p dx \\ &= \sum_{i\in\mathbb{N}_{N-1}} \int_{I_i} |(f_n^{(q,\alpha)}(L_i^{-1}(x)) - \Phi_{q,n}(f;L_i^{-1}(x)))\alpha_i(L_i^{-1}(x))|^p dx \\ &= \sum_{i\in\mathbb{N}_{N-1}} a_i \int_I |(f_n^{(q,\alpha)}(t) - \Phi_{q,n}(f;t))\alpha_i(t)|^p dt \\ &\le \sum_{i\in\mathbb{N}_{N-1}} a_i\|\alpha_i\|_\infty^p \int_I |(f_n^{(q,\alpha)}(t) - \Phi_{q,n}(f;t))|^p dt \\ &= \sum_{i\in\mathbb{N}_{N-1}} a_i\|\alpha_i\|_\infty^p \|f_n^{(q,\alpha)} - \Phi_{q,n}(f)\|_p^p. \end{aligned}$$

Taking pth root in both sides, we have

$$\begin{aligned} \|f_n^{(q,\alpha)} - f\|_p &\le \left[\sum_{i\in\mathbb{N}_{N-1}} a_i\|\alpha_i\|_\infty^p\right]^{\frac{1}{p}} \|f_n^{(q,\alpha)} - \Phi_{q,n}(f)\|_p, \\ &\le \left[\sum_{i\in\mathbb{N}_{N-1}} a_i\|\alpha_i\|_\infty^p\right]^{\frac{1}{p}} [\|f_n^{(q,\alpha)} - f\|_p + \|f - \Phi_{q,n}(f)\|_p], \end{aligned}$$

and further implication gives

$$\|f_n^{(q,\alpha)} - f\|_p \le \frac{\left[\sum_{i\in\mathbb{N}_{N-1}} a_i\|\alpha_i\|_\infty^p\right]^{\frac{1}{p}}}{1 - \left[\sum_{i\in\mathbb{N}_{N-1}} a_i\|\alpha_i\|_\infty^p\right]^{\frac{1}{p}}} \|f - \Phi_{q,n}(f)\|_p. \tag{19}$$

A similar calculation as in Proposition 2, we obtain

$$\|f_n^{(q,\alpha)} - f\|_\infty \le \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty} \|f - \Phi_{q,n}(f)\|_\infty. \tag{20}$$

From the last two inequalities, we get the desired result. ∎

Theorem 7. *The (q,α)-Kantorovich-Bernstein fractal operator $\mathcal{F}_{\Delta,\Phi_{q,n}}^{(q,\alpha)} : \mathcal{L}^p(I) \mapsto \mathcal{L}^p(I), 1\le p\le\infty,\ n\in\mathbb{N}$ defined by $\mathcal{F}_{\Delta,\Phi_{q,n}}^{(q,\alpha)}(f) = f_n^{(q,\alpha)}$ is linear and bounded.*

Proof. Let f and g be in $\mathcal{L}^p(I)$ and λ_1, λ_2 be real scalars. The functional equations for the corresponding (q, α)-Kantorovich-Bernstein fractal functions are given by

$$f_n^{(q,\alpha)}(x) = f(x) + \alpha_i(L_i^{-1}(x))(f_n^{(q,\alpha)}(L_i^{-1}(x)) - \Phi_{q,n}(f; L_i^{-1}(x))),$$

$$g_n^{(q,\alpha)}(x) = g(x) + \alpha_i(L_i^{-1}(x))(g_n^{(q,\alpha)}(L_i^{-1}(x)) - \Phi_{q,n}(g; L_i^{-1}(x))) \quad \forall x \in I_i, \quad i \in \mathbb{N}_{N-1}.$$

Thus, we can write

$$\begin{aligned}(\lambda_1 f_n^{(q,\alpha)} + \lambda_2 g_n^{(q,\alpha)})(x) = (\lambda_1 f + \lambda_2 g)(x) + \alpha_i(L_i^{-1}(x))[(\lambda_1 f_n^{(q,\alpha)} + \lambda_2 g_n^{(q,\alpha)})(L_i^{-1}(x)) \\ - \Phi_{q,n}(\lambda_1 f + \lambda_2 g; L_i^{-1}(x))]\end{aligned} \tag{21}$$

from which we obtain that $\lambda_1 f_n^{(q,\alpha)} + \lambda_2 g_n^{(q,\alpha)}$ is a fixed point of the operator

$$(Th)(x) = (\lambda_1 f + \lambda_2 g)(x) + \alpha_i(L_i^{-1}(x))(h - \Phi_{q,n}(\lambda_1 f + \lambda_2 g; L_i^{-1}(x)).$$

Now using the uniqueness of fixed point, we get

$$\mathcal{F}_{\Delta,\Phi_{q,n}}^{(q,\alpha)}(\lambda_1 f + \lambda_2 g) = \lambda_1 f_n^{(q,\alpha)} + \lambda_2 g_n^{(q,\alpha)} = \lambda_1 \mathcal{F}_{\Delta,\Phi_{q,n}}^{(q,\alpha)}(f) + \lambda_2 \mathcal{F}_{\Delta,\Phi_{q,n}}^{(q,\alpha)}(g).$$

Again with help of (19) and (20), we have

$$\begin{aligned}\|\mathcal{F}_{\Delta,\Phi_{q,n}}^{(q,\alpha)}(f)\|_p &= \|f_n^{(q,\alpha)}\|_p \\ &\le \|f_n^{(q,\alpha)} - f\|_p + \|f\|_p \\ &\le \frac{R}{1-R}\|f - \Phi_{q,n}(f)\|_p + \|f\|_p \\ &\le \frac{R}{1-R}\|Id - \Phi_{q,n}\|_p\|f\|_p + \|f\|_p,\end{aligned} \tag{22}$$

where

$$R = \begin{cases} \left[\sum_{i\in\mathbb{N}_{N-1}} a_i\|\alpha_i\|_\infty^p\right]^{\frac{1}{p}}, & \text{for } 1 \le p < \infty, \\ \|\alpha\|_\infty, & \text{for } p = \infty. \end{cases} \tag{23}$$

Since $\|Id - \Phi_{q,n}\|_p \to 0$ as $n \to \infty$, so for given $\epsilon = 1$, there exists $M \in \mathbb{N}$ such that

$$\|Id - \Phi_{q,n}\|_p < 1 \quad \forall\, n > M.$$

Consider $\eta = \max\{\|Id - \Phi_{q,1}\|_p, \|Id - \Phi_{q,2}\|_p, \dots, \|Id - \Phi_{q,M}\|_p, 1\}$. Then from (22) we get

$$\|\mathcal{F}_{\Delta,\Phi_{q,n}}^{(q,\alpha)}\| \le 1 + \frac{R}{1-R}\eta,$$

which implies $\mathcal{F}_{\Delta,\Phi_{q,n}}^{(q,\alpha)}$ is bounded operator for each $n \in \mathbb{N}$. ∎

Theorem 8. *Consider a scaling function which satisfies*

$$\left[\sum_{i\in\mathbb{N}_{N-1}} a_i\|\alpha_i\|_\infty^p\right]^{\frac{1}{p}} < \min\{1, \|\Phi_{q,n}\|^{-1}\}, \quad \text{if } 1 \le p < \infty,$$

$$\|\alpha\|_\infty < \min\{1, \|\Phi_{q,n}\|^{-1}\}, \quad \text{if } p = \infty.$$

Then the corresponding fractal operator is bounded below. In particular, $\mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}$ is injective and has a closed range.

Proof. From the reverse triangle inequality and the proof of Theorem 6, we obtain

$$\begin{aligned} \|f\|_p - \|f_n^{(q,\alpha)}\|_p &\le \|f - f_n^{(q,\alpha)}\|_p \\ &\le R\|f_n^{(q,\alpha)} - \Phi_{q,n}(f)\|_p \\ &\le R\|f_n^{(q,\alpha)}\|_p + R\|\Phi_{q,n}\|\|f\|_p \\ \Rightarrow \|f\|_p &\le \frac{1+R}{1-R\|\Phi_{q,n}\|}\|f_n^{(q,\alpha)}\|_p. \end{aligned} \tag{24}$$

Since $\|\Phi_{q,n}\|^{-1} > R$, the operator $\mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}$ is bounded below and so injective. Now to prove $\mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}$ has a closed range, let $f^{(q,\alpha)}_{n,m}$ be a sequence in $\mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}(\mathcal{L}^p(I))$ such that $f^{(q,\alpha)}_{n,m} \to \tilde{f}$, and thus, $f^{(q,\alpha)}_{n,m}$ is a Cauchy sequence. Now

$$\|f_m - f_r\|_p \le \frac{1+R}{1-R\|\Phi_{q,n}\|}\|f^{(q,\alpha)}_{m,n} - f^{(q,\alpha)}_{r,n}\|_p,$$

which shows that $\{f_m\}$ is a Cauchy sequence in $\mathcal{L}^p(I)$. Since $\mathcal{L}^p(I)$ is a complete metric space, there exists $f \in \mathcal{L}^p(I)$ such that $f_m \to f$. Using the continuity of $\mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}$, we have $\tilde{f} = \mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}(f) = f_n^{(q,\alpha)}$. ■

6 | APPROXIMATION BY KANTOROVICH-BERNSTEIN FRACTAL FUNCTIONS

Denote $\Lambda := \{\lambda_i\}_{i=1}^{+\infty}$, $\lambda_i \neq \lambda_j$ if $i \neq j$, $\lambda_i \in \mathbb{R}^+$, $\lambda_0 = 0$. The collection $\Lambda_m = \{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_m}\}$ is called a finite Müntz system. The linear span of Λ_m is known as Müntz space and denoted by $M_m(\Lambda)$. Let $I = [a,b], a > 0$ and $\Delta := \{x_1, \dots, x_N\}$ be a partition of I satisfying $a = x_1 < \dots < x_N = b$. Choose the scaling function $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{N-1}) \in (\mathcal{L}^\infty(I))^{N-1}$ as per the prescription given in Theorem 5. We know that $\Phi_{q,n} : \mathcal{L}^p(I) \mapsto \mathcal{L}^p(I)$ is a bounded linear map and the Müntz monomial $x^{\lambda_i} \in \mathcal{L}^p(I)$ even if $\lambda_i > \frac{-1}{p}$. Therefore, we can define the (q,α)-Kantorovich-Bernstein fractal Müntz monomial $(x^{\lambda_i})_n^{(q,\alpha)} := \mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}(x^{\lambda_i})$.

Definition 2. A (q,α)-Kantorovich-Bernstein fractal Müntz polynomial is a finite linear combination of the functions $(x^{\lambda_i})_n^{(q,\alpha)}$, where $\lambda_i \in \Lambda$, $i \in \mathbb{N}$, and $\alpha \in (\mathcal{L}^\infty(I))^{N-1}$ satisfies the condition of Theorem 5. In particular, when $\alpha = \mathbf{0}$, this linear combination is called quantum Bernstein Müntz polynomial.

Let $S = \{(x^{\lambda_i})_n^{(q,\alpha)} : i, n \in \mathbb{N}\}$. The set

$$M^{(q,\alpha)}(\Lambda) := Span(S)$$

is defined as the quantum Bernstein fractal Müntz space associated with Λ. We need the following definition in the sequel:

Definition 3 (39). A set A is fundamental in a normed linear space B if the family of linear combinations of elements of A is a dense set of B.

Theorem 9 (Quantum fractal version of first Müntz theorem). *Let Δ be a partition of $I = [a,b], b > 0$. If the scaling vector α is chosen according to Theorem 5, then the system S restricted to values λ_i such that $-\frac{1}{2} < \lambda_i \to \infty$ is fundamental in $\mathcal{L}^2(I)$, whenever $\sum_{\lambda_i \neq 0} \frac{1}{\lambda_i} = +\infty$.*

Proof. Let $g \in \mathcal{L}^2(I)$ and $\epsilon > 0$ be given. From classical Müntz's first theorem (see for instance Reference 39), it is known that the set of functions $\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$, where $-\frac{1}{2} < \lambda_i \to \infty$ is fundamental in the least-square norm if and only if $\sum_{\lambda_i \neq 0} \frac{1}{\lambda_i} = +\infty$.

Thus, for $\epsilon/2 > 0$, there exists a Müntz polynomial $q_m \in Span\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ such that

$$\|g - q_m\|_2 < \frac{\epsilon}{2}. \tag{25}$$

With the scaling function α, we construct the (q,α)-Kantorovich-Bernstein fractal Müntz polynomial as $(q_m)_n^{(q,\alpha)} = \mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}(q_m)$ by using the linearity of $\mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}$. Since $\|q_m - \Phi_{q,n}(q_m)\|_2 \to 0$ as $n \to \infty$, there exists $M_1 \in \mathbb{N}$ such that

$$\|q_m - \Phi_{q,n}(q_m)\|_2 < \frac{\epsilon\left[1 - \sqrt{\sum_{i\in\mathbb{N}_{N-1}} a_i \|\alpha_i\|_\infty^2}\right]}{2\sqrt{\sum_{i\in\mathbb{N}_{N-1}} a_i \|\alpha_i\|_\infty^2}} \quad \text{for } n > M_1. \tag{26}$$

Using (26) in (19), we obtain

$$\|(q_m)_n^{(q,\alpha)} - q_m\|_2 \le \frac{\sqrt{\sum_{i\in\mathbb{N}_{N-1}} a_i \|\alpha_i\|_\infty^2}}{1 - \sqrt{\sum_{i\in\mathbb{N}_{N-1}} a_i \|\alpha_i\|_\infty^2}} \|q_m - \Phi_{q,n}(q_m)\|_2 < \frac{\epsilon}{2} \quad \text{for } n > M_1. \tag{27}$$

Combining (25) and (27), we have

$$\|g - (q_m)_n^{(q,\alpha)}\|_2 \le \|g - q_m\|_2 + \|(q_m)_n^{(q,\alpha)} - q_m\|_2 < \epsilon \quad \text{for } n > M_1.$$

Consequently $(q_m)_n^{(q,\alpha)} \in M^{(q,\alpha)}(\Lambda)$ approximates to g in $\mathcal{L}^2$-norm and the set considered is fundamental in $\mathcal{L}^2(I)$. ∎

Corollary 1. *The system S is complete in $\mathcal{L}^2(I)$ if the scaling vector α is chosen according to the prescription of Theorem 5, $-\frac{1}{2} < \lambda_i \to \infty$ and $\sum_{\lambda_i \neq 0} \frac{1}{\lambda_i} = +\infty$.*

Proof. In the above theorem, we have proved that $\{(x^{\lambda_i})_n^{(q,\alpha)} : i, n \in \mathbb{N}\}$ where λ_i satisfy the conditions described is fundamental in the normed linear space $\mathcal{L}^2(I)$. According to Banach's theorem (see for instance Reference 40), the system S is complete. ∎

We can generalize the above results for any fundamental system of $\mathcal{L}^p(I), 1 \le p < \infty$. The proof follows similar lines and hence it is omitted.

Theorem 10. *Let Δ be a partition of $I = [a, b], a > 0$ and the scaling vector α be chosen according to the prescription of Theorem 5. If the system $\{f_j : j \in \mathbb{N}\}$ is fundamental in $\mathcal{L}^p(I), 1 \le p < \infty$, then the corresponding quantum Bernstein fractal system $\{(f_j^{\lambda_i})_n^{(q,\alpha)} : i, j, n \in \mathbb{N}\}$ is also fundamental whenever $-\frac{1}{p} < \lambda_i \to \infty$ and $\sum_{\lambda_i \neq 0} \frac{1}{\lambda_i} = +\infty$.*

Now, we will state the full Müntz theorem in $L^p[0,1], 1 \le p \le \infty$ for quantum fractal Bernstein Müntz polynomials. The proof follows similar steps as described in Theorem 9 with proper choice of classical Müntz polynomial, that is, the exponents satisfy the condition prescribed by Borwein and Erdélyi.[41]

Theorem 11. *Let $1 \le p \le \infty$ and $\Delta := 0 = x_0 < x_1 < \dots x_N = 1$ be a partition of $I = [0,1]$. Let $\Lambda := \{\lambda_i\}_{i=0}^\infty$ be a sequence of distinct real numbers greater than $\frac{-1}{p}$, and such that $\sum_{i=0}^\infty \frac{\lambda_i + \frac{1}{p}}{(\lambda_i + \frac{1}{p})^2 + 1} = \infty$. Then, the system $\{(x^{\lambda_i})_n^{(q,\alpha)} : i, n \in \mathbb{N}\}$ is fundamental in $\mathcal{L}^p(I)$.*

7 | CONCLUSION

In the present article, we have introduced a new approximation method using q-Bernstein polynomial as the base function in the structure of fractal interpolants. For a given function $f \in C(I)$, the convergence of the sequence of the quantum fractal functions toward f does not need any further condition on the scaling functions so that these approximants can be smooth or nonsmooth depending on the norm of the scaling functions. The shape of the proposed fractal approximants depends on the free variable $q \in (0,1)$ apart from the scaling functions. Hence, for the given continuous function f, the proposed quantum fractal approximants provide a large number of approximants than that would be obtained by the existing fractal approximants. It is observed that the multivalued quantum fractal operator $\mathcal{F}^{(q,\alpha)} : C(I) \rightrightarrows C(I)$ is not linear. The (q,α)-Kantorovich-Bernstein fractal functions in $\mathcal{L}^p$ spaces are developed and their approximation properties (quantum analogue of Müntz theorems) are studied.

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ORCID

M. A. Navascués https://orcid.org/0000-0003-4847-0493
M. V. Sebastián https://orcid.org/0000-0002-0477-835X

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AUTHOR BIOGRAPHIES

N. Vijender received the master in Science in Mathematics from National Institute of Technology (NIT) Warangal, Telangana in 2008. Subsequently, he received PhD from Indian Institute of Technology (IIT) Madras in 2014. He worked as an assistant professor at Vellore Institute of Technology (VIT) Chennai during 2004-2018. Since 2019, he has been working as a faculty at IIIT Nagpur. His current research interests include fractal approximation, shape preserving fractal splines, and fractal numerical methods.

A.K.B. Chand received the master in Science and master in Philosophy in Mathematics from Utkal University, Bhubaneswar, Odisha in 1996 and 1997, respectively. Then, he received PhD from IIT Kanpur in 2005. He worked as assistant professor in BITS-Pilani Goa campus prior to his postdoctoral position at University of Zaragoza, Spain in 2007. Since 2008, he has been working as a faculty at IIT Madras and currently, he is a professor. His current research interests include fractal interpolation functions/surfaces, shape preserving fractals, approximation by fractal functions, computer-aided geometric design, wavelets, and fractal signal/image processing.

M.A. Navascués is professor of the Engineering and Architecture School of the University of Zaragoza (Spain). She completed her doctorate at the same University. She has been visitor professor at several foreign Universities. She leads two research groups, in the University of Zaragoza and in the Indian Institute of Technology of Madras. Since November 2015 until March 2017 she was the secretary of the Real Sociedad Matemática Española (Royal Spanish Mathematical Society). Now she is territorial delegate of the same society.

M.V. Sebastián, PhD in Mathematics, professor at the Centro Universitario de la Defensa de Zaragoza (Academia General Militar), attached to the University of Zaragoza. Research lines: approximation and adjustment of curves, fractal interpolation, chaos theory, nonlinear dynamics and processing and quantification of experimental signals (in particular electroencephalographic signals). Member of the University Institute of Mathematics and Applications Research (IUMA) and of the research group "Mathematical Physics and Fractal Geometry" (recognized by the Government of Aragón).

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