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Moore–Penrose inverse positivity of interval matrices

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ABSTRACT

For $A, B \in \mathbb{R}^{m\times n}$, let $J = [A, B]$ be the set of all matrices C such that $A \leqslant C \leqslant B$, where the order is component wise. Krasnosel'skij et al. [9] and Rohn [11] have shown that if A and B are invertible with $A^{-1} \geqslant 0$ and $B^{-1} \geqslant 0$, then every $C \in J$ is invertible with $C^{-1} \geqslant 0$. In this article, we present certain extensions of this result to the singular case, where the nonnegativity of the usual inverses is replaced by the nonnegativity of the Moore–Penrose inverse.

1. Introduction

A real $n \times n$ matrix A is called *monotone* if $Ax \geqslant 0 \implies x \geqslant 0$. Here, $y \geqslant 0$ for $(y_1, y_2, \ldots, y_n)^T = y \in \mathbb{R}^n$ means that $y_i \geqslant 0$ for all $i = 1, 2, \ldots, n$. This notion was introduced by Collatz, who showed that A is monotone if and only if A^{-1} exists and $A^{-1} \geqslant 0$, where the latter denotes that all the entries of A^{-1} are nonnegative. The book by Collatz [6] has details of how monotone matrices arise naturally in the study of finite difference approximation methods for certain elliptic partial differential equations. The problem of characterizing monotone (also referred to as inverse-positive) matrices has been extensively studied in the literature. The books by Berman and Plemmons [5] and Varga [14] give an excellent account of many of these characterizations.

Much effort has been devoted to characterizing inverse-positive matrices in terms of the so-called splittings of the matrix concerned. For a real $n \times n$ matrix A, a decomposition $A = U - V$ is called a splitting, if U is invertible. Associated with the splitting, one studies convergence of the iterative method $x^{k+1} = U^{-1}Vx^k + U^{-1}b,\ k = 0, 1, 2, \ldots$, for numerically solving the linear system of

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equations $Ax = b,\ b \in \mathbb{R}^n$. It is well known that this iterative scheme converges to a solution of $Ax = b$ if and only if $\rho(U^{-1}V) < 1$, for any initial vector x^0, where $\rho(M)$ will denote the spectral radius of the square matrix M. Standard iterative methods like the Jacobi, Gauss–Seidel and successive over-relaxation methods arise from different choices of U and V. In this regard, Ortega and Rheinboldt [10] proposed the notion of a weak regular splitting: $A = U - V$ is called a weak regular splitting if U is invertible, $U^{-1} \geqslant 0$ and $U^{-1}V \geqslant 0$. They showed that, $A^{-1} \geqslant 0$ if and only if $\rho(U^{-1}V) < 1$, for any weak regular splitting $A = U - V$. We refer to [14] for a proof.

In what follows (Theorem 3.3), first we present a generalization of this result for the Moore–Penrose inverse. Even though such a generalization is already available in the literature [3, Theorem 3], we believe that our proof is simpler and closely follows the proof of Varga, mentioned above. In this article, one of the main objects of study are interval matrices. Following [9], we define a *bilateral interval* J as $J = [A, B] = \{C : A \leqslant C \leqslant B\}$ for $A \in \mathbb{R}^{m\times n},\ B \in \mathbb{R}^{m\times n}$ and $A \leqslant B$. If $J = (-\infty, B]$ (so that $C \in J$ if and only if $C \leqslant B$), then J will be called a *unilateral interval*. For a unilateral interval, the following result was proved by Krasnoselskij et al. [9, Theorem 25.4]. The original result holds even for Banach spaces. We are only concerned with the finite dimensional version. $int(\mathbb{R}^n_+)$ denotes the set of all interior points of $\mathbb{R}^n_+$.

Theorem 1.1. *Let $B, C \in \mathbb{R}^{n\times n},\ C \leqslant B$, B being invertible with $B^{-1} \geqslant 0$. Then $C^{-1} \geqslant 0$ if and only if $int(\mathbb{R}^n_+) \cap C\mathbb{R}^n_+ \neq \emptyset$.*

Our next main result is a generalization of the result above, for singular matrices, even rectangular. This is an extension to the case of Moore–Penrose inverse, presented in Theorem 3.4.

Next, we turn to a result for bilateral intervals, proved by Rohn (see also, [9, Theorem 25.6]). The matrices $J_c = \frac{1}{2}(B + A)$ and $\Delta = \frac{1}{2}(B - A)$ are referred to as the center and the radius of the interval matrix J, respectively. Then $\Delta \geqslant 0$, $A = J_c - \Delta$, $B = J_c + \Delta$ and an alternative description of the interval J is then given by $J = [J_c - \Delta, J_c + \Delta]$. J is said to be *regular* if C^{-1} exists for all $C \in J$ and *inverse positive* if $C^{-1} \geqslant 0$ for each $C \in J$. Rohn characterized inverse positivity of bilateral interval matrices in the following result:

Theorem 1.2 [11, Theorem 1]**.** *Let $J = [A, B]$. Then the following statements are equivalent:*

(a) *J is inverse positive.*
(b) *$A^{-1} \geqslant 0$ and $B^{-1} \geqslant 0$.*
(c) *$B^{-1} \geqslant 0$ and $\rho(B^{-1}(B - A)) < 1$.*
(d) *$B^{-1} \geqslant 0$ and J is regular.*

Our third main result presents an extension of this result for the Moore–Penrose inverse. This is done in Theorem 3.5. The applicability of Theorem 3.5 hinges upon the nonemptiness of a certain set of matrices K. Theorem 3.8 presents a sufficient condition under which $K \neq \emptyset$. Section 3 also presents a couple of other results related to interval matrices.

The article is organized as follows: This introductory section is followed by the section on preliminaries. The last section deals with the case of Moore–Penrose inverse positivity. Extensions of the results of this article to the case of infinite dimensional spaces and other types of generalized inverses will be studied in future.

2. Notation, definitions and preliminary results

All matrices will have real entries. $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ matrices over the reals. For $A \in \mathbb{R}^{m\times n}$, we denote the transpose of A, the range space of A and null space of A by $A^t,\ R(A)$ and $N(A)$, respectively.

For a given $A \in \mathbb{R}^{m\times n}$, the unique matrix $X \in \mathbb{R}^{n\times m}$ satisfying $AXA = A,\ XAX = X,\ (AX)^t = AX$ and $(XA)^t = XA$ is called the Moore–Penrose inverse of A and is denoted by $A^\dagger$. For complementary subspaces L and M of $\mathbb{R}^n$, the (not necessarily orthogonal) projection of $\mathbb{R}^n$ on L along M will be denoted by $P_{L,M}$. If, in addition, L and M are orthogonal then we denote this by P_L. Some of the well known properties of $A^\dagger$ which will be frequently used, are [1]: $R(A^t) = R(A^\dagger)$; $N(A^t) = N(A^\dagger)$; $AA^\dagger = P_{R(A)}$; $A^\dagger A = P_{R(A^t)}$. In particular, if $x \in R(A^t)$ then $x = A^\dagger Ax$.

If A and B are square invertible matrices, then $(AB)^{-1} = B^{-1}A^{-1}$. However, for a generalized inverse this is not always the case. The following result presents a characterization for the "reverse order law" to hold for the case of the Moore–Penrose inverse.

Theorem 2.1 [8, Theorem 1]**.** *If A and B are arbitrary rectangular matrices such that AB is defined, then,* $(AB)^\dagger = B^\dagger A^\dagger$ *if and only if* $BB^tA^t = A^\dagger ABB^tA^t$ *and* $A^tAB = BB^\dagger A^tAB$.

We will need the following particular case of Theorem 2.1.

Corollary 2.1. *If A and B are arbitrary matrices such that B is invertible and AB is defined, then,* $(AB)^\dagger = B^{-1}A^\dagger$ *if and only if* $BB^tA^t = A^\dagger ABB^tA^t$.

Recall that the spectral radius $\rho(A)$ of a matrix $A \in \mathbb{R}^{n\times n}$ is defined to be the maximum of the moduli of all the eigen values of A. Of course, A may have complex eigen values.

Lemma 2.1. *Let* $A, B \in \mathbb{R}^{n\times n}$ *satisfy* $A \geqslant B \geqslant 0$. *Then* $\rho(A) \geqslant \rho(B)$.

The following result gives an estimate for the spectral radius of a matrix. This will, again, be used in one of the proofs to follow.

Theorem 2.2 [9, Theorem 16.2]**.** *For* $A \in \mathbb{R}^{n\times n}$ *suppose that the inequality* $Ax \leqslant \delta x$ *holds, for some* $x > 0$ *(meaning that all the coordinates of x are positive). Then* $\rho(A) \leqslant \delta$.

Let us recall that if $A \in \mathbb{R}^{n\times n}$ satisfies $\rho(A) < 1$, then $I - A$ is invertible. The next result is frequently used in the study of nonnegative matrices. It gives a sufficient condition under which $(I - A)^{-1}$ is nonnegative.

Theorem 2.3 [14, Theorem 3.16]**.** *Let* $A \in \mathbb{R}^{n\times n}$. *Then* $\rho(A) < 1$ *if and only if* $(I - A)^{-1}$ *exists and* $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$. *If, in addition,* $A \geqslant 0$, *then* $(I - A)^{-1} \geqslant 0$.

3. Moore–Penrose inverse positivity

In this section, we study the positivity of the Moore–Penrose inverse of unilateral and bilateral intervals. Central to the discussion is the notion of a proper splitting, which we discuss next. A decomposition $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a proper splitting if $R(A) = R(U)$ and $N(A) = N(U)$. This notion was introduced and studied in [3] with the purpose of extending classical iterative methods, especially applicable to the case of singular (often rectangular) matrices. The first result below collects some properties of such a splitting.

Theorem 3.1 [3, Theorem 1]**.** *Let* $A = U - V$ *be a proper splitting of* $A \in \mathbb{R}^{m\times n}$. *Then*

(a) $A = U(I - U^\dagger V)$,
(b) $I - U^\dagger V$ *is nonsingular,*
(c) $A^\dagger = (I - U^\dagger V)^{-1}U^\dagger$ *and*
(d) $A^\dagger b$ *is the unique solution to the system* $x = U^\dagger Vx + U^\dagger b$, *for any* $b \in \mathbb{R}^m$.

Remark 3.1. We observe that if $A = U - V$ is a proper splitting of A, then $R(V) \subseteq R(A)$ and that $A^t = U^t - V^t$ is a proper splitting of A^t. In that case, we have $R(V^t) \subseteq R(A^t)$. Thus, if $A = U - V$ is a proper splitting of A, then $AA^\dagger = P_{R(A)} = P_{R(U)} = UU^\dagger$ and $A^\dagger A = P_{R(A^t)} = P_{R(U^t)} = U^\dagger U$.

Thus, $UU^{\dagger}V = P_{R(U)}V = P_{R(A)}V = V$, since $R(V) \subseteq R(A)$. Also, $VU^{\dagger}U = V(U^{\dagger}U)^t = (U^{\dagger}UV^t)^t = (P_{R(U^t)}V^t)^t = (P_{R(A^t)}V^t)^t = (V^t)^t = V$, since $R(V^t) \subseteq R(A^t)$.

It is well known that any consistent linear system of equations $Ax = b$ for $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ is solved in practice by iterative methods. Broadly, these methods have the form $x^{k+1} = Hx^k + c$, for $H \in \mathbb{R}^{n\times n}$ and $c \in \mathbb{R}^n$. Then, the convergence of the sequence x^{k+1} (for any initial vector x^0) is guaranteed by the spectral radius condition $\rho(H) < 1$. H is called the iteration matrix of the method. For a proper splitting given as above, we have $H = U^{\dagger}V$ (and $c = U^{\dagger}b$). The next result gives a set of sufficient conditions under which $\rho(H) < 1$ can be guranteed. We will use the following notion.

Definition 3.1. A decomposition $A = U - V$ is called a weak pseudo regular splitting if it is a proper splitting such that $U^{\dagger} \geqslant 0$ and $U^{\dagger}V \geqslant 0$.

Theorem 3.2 [3, Theorem 3]**.** *Let $A = U - V$ be a weak pseudo regular splitting of A. Then the following statements are equivalent:*

(a) $A^{\dagger} \geqslant 0$.
(b) $A^{\dagger}V \geqslant 0$.
(c) $\rho(U^{\dagger}V) < 1$.

In the next result we present another proof of the equivalence of (a) and (c). This proof is an adaptation of the proof of Varga [14, Theorem 3.37]. For a version in infinite dimensional spaces, for the nonsingular case, see [13].

Theorem 3.3. *Let $A = U - V$ be a weak pseudo regular splitting of $A \in \mathbb{R}^{m\times n}$. Then $A^{\dagger} \geqslant 0$ if and only if $\rho(U^{\dagger}V) < 1$.*

Proof. Let $C = U^{\dagger}V$. Then $C \geqslant 0$. Also, $CU^{\dagger}U = U^{\dagger}VU^{\dagger}U = U^{\dagger}V = C$, since (as was proved in Remark 3.1), we have $VU^{\dagger}U = V$. In general, for $k \geqslant 1$ we have $C^{k+1}U^{\dagger}U = C^{K+1}$. From (a) and (c) of Theorem 3.1, we have $A = U(I - C)$ and $A^{\dagger} = (I - C)^{-1}U^{\dagger}$. If $\rho(C) < 1$, then by Theorem 2.3, $(I - C)^{-1}$ exists and $(I - C)^{-1} \geqslant 0$ so that $A^{\dagger} = (I - C)^{-1}U^{\dagger} \geqslant 0$, where we have used the fact that $U^{\dagger} \geqslant 0$.

Conversely, suppose that $A^{\dagger} \geqslant 0$. Set $B_k = (I + C + C^2 + C^3 + \ldots + C^k)U^{\dagger}$ for any positive integer k. Then $B_k \geqslant 0$ and $B_k \leqslant B_{k+1}$, since $C \geqslant 0$. Using $U^{\dagger} = (I - C)A^{\dagger}$ it then follows that $B_k = (I - C^{k+1})A^{\dagger}$. Again, since $C \geqslant 0$ and $A^{\dagger} \geqslant 0$, it follows that $B_k \leqslant A^{\dagger}$. Hence the sequence $\{B_k\}$ is a monotonically increasing sequence, which is bounded above. Hence $\{B_n\}$ is convergent with respect to any matrix norm $\| \cdot \|$. Also, $B_{k+1}U - B_kU = C^{k+1}U^{\dagger}U = C^{k+1}$. So, $\| B_{k+1}U - B_kU \| = \| C^{k+1} \| \leqslant \| B_{k+1} - B_k \| \| U \|$. We conclude that C^{k+1} converges to the zero matrix. It now follows that $\rho(U^{\dagger}V) < 1$. □

For unilateral intervals, we have the following result. This is an extension of Theorem 1.1 mentioned in Section 1. We observe that the proof can be carried over verbatim to infinite dimensional spaces.

Theorem 3.4. *Let $B, C \in \mathbb{R}^{m\times n}$, $R(B) = R(C)$, $N(B) = N(C)$, $C \leqslant B$ and $B^{\dagger} \geqslant 0$. Then $C^{\dagger} \geqslant 0$ if and only if $int(\mathbb{R}^m_+) \cap \{C\mathbb{R}^n_+ + N(C^t)\} \neq \emptyset$.*

Proof. Let $C^{\dagger} \geqslant 0$. Then $C^{\dagger}(\mathbb{R}^m_+) \subseteq \mathbb{R}^n_+$, and so, $CC^{\dagger}\mathbb{R}^m_+ \subseteq C\mathbb{R}^n_+$. For $x \in \mathbb{R}^m_+$, let $x = x^1 + x^2$, where $x^1 \in R(C)$ and $x^2 \in R(C)^{\perp} = N(C^t)$. Then $x^1 = CC^{\dagger}x$ and so, $x = CC^{\dagger}x + x^2 \in C\mathbb{R}^n_+ + N(C^t)$.

Conversely, suppose that $int(\mathbb{R}^m_+) \cap \{C\mathbb{R}^n_+ + N(C^t)\} \neq \emptyset$. Since $C \leqslant B$, there exists $T \geqslant 0$ such that $C = B - T$. By the hypotheses, we have $R(B) = R(C)$, $N(B) = N(C)$ and $B^{\dagger} \geqslant 0$. Also, $B^{\dagger}T \geqslant 0$ and so, $C = B - T$ is a weak pseudo regular splitting. We show that $\rho(B^{\dagger}T) < 1$. It would then follow from Theorem 3.3 that $C^{\dagger} \geqslant 0$.

Let $x \in \mathbb{R}^n_+$ and $z \in N(C^t)$ such that $Cx+z \in int(\mathbb{R}^m_+)$. Since $C \leqslant B$, it follows that $Bx+z \in int(\mathbb{R}^m_+)$. Since $Bx+z$ and $Cx+z$ are positive, there exist $\epsilon > 0$ such that $\epsilon(Bx+z) \leqslant (Cx+z)$, so that $(B-C)x \leqslant (1-\epsilon)(Bx+z)$. We have $z \in N(C^t) = N(B^t) = N(B^\dagger)$, so that $B^\dagger z = 0$. Then, $TB^\dagger(Bx+z) = TB^\dagger Bx = (B-C)B^\dagger Bx = (BB^\dagger B - CB^\dagger B)x = (B-C)x$, using the fact that $B^\dagger B = C^\dagger C$. Then $TB^\dagger(Bx+z) \leqslant (1-\epsilon)(Bx+z)$. Also $Bx+z \in int(\mathbb{R}^m_+)$. Hence $\rho(TB^\dagger) \leqslant 1-\epsilon < 1$, by Theorem 2.2. As mentioned before, it now follows that $C^\dagger \geqslant 0$. □

Remark 3.2. In the above theorem, the conditions $R(B) = R(C)$ and $N(C) = N(B)$ both are indispensable.

Let $B = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$. Then $B^\dagger = \begin{pmatrix} 1/4 & 1/4 \\ 0 & 0 \end{pmatrix} \geqslant 0$ and $C^\dagger = \begin{pmatrix} 1/4 & 1/4 \\ -1/4 & -1/4 \end{pmatrix} \ngeqslant 0$.

Here, $N(C) \neq N(B)$, $R(C) = R(B)$ and $int(\mathbb{R}^2_+) \cap (C\mathbb{R}^2_+ + N(C^t)) \neq \emptyset$.

Corollary 3.1. *Suppose that the hypotheses of Theorem 3.4 hold and that $C^\dagger \geqslant 0$. Then $B^\dagger \leqslant C^\dagger$.*

Proof. We have $B^\dagger \geqslant 0$, $C^\dagger \geqslant 0$. So, $C \leqslant B$ implies $C^\dagger CB^\dagger \leqslant C^\dagger BB^\dagger$. The proof is complete by observing that $C^\dagger CB^\dagger = B^\dagger BB^\dagger = B^\dagger$ and $C^\dagger BB^\dagger = C^\dagger CC^\dagger = C^\dagger$. □

Now, we propose a notion of regularity for interval matrices, appropriate enough for singular matrices.

Definition 3.2. The bilateral interval matrix $J = [A, B]$ is called range kernel regular, if $R(A) = R(B)$ and $N(A) = N(B)$.

The proposed generalization of Theorem 1.2 to the Moore–Penrose inverse is only applicable to a subset K of J, which we define as

$$K = \{C \in J : R(C) = R(B) = R(A) \text{ and } N(C) = N(B) = N(A)\}.$$

Later, in Theorem 3.8, we present sufficient conditions under which K is shown to be nonempty.

Now, we prove the aforementioned extension of Rohn's result (Theorem 1.2). Our proof is completely different from the proof of Rohn and relies solely on (the idea of proper splittings and) Theorem 3.3.

Theorem 3.5. *Let $J = [A, B]$ be range kernel regular. Then the following statements are equivalent:*

(a) $C^\dagger \geqslant 0$ *whenever* $C \in K$.
(b) $A^\dagger \geqslant 0$ *and* $B^\dagger \geqslant 0$.
(c) $B^\dagger \geqslant 0$ *and* $\rho(B^\dagger(B-A)) < 1$.

Proof

(a) ⇒ (b): Follows from the definition.
(b) ⇒ (c): Set $U = B$ and $V = B - A$. Then $A = U - V$ with $R(U) = R(B) = R(A)$ and $N(U) = N(B) = N(A)$. Thus $A = U - V$ is a proper splitting. Also, $U^\dagger V = B^\dagger(B-A) \geqslant 0$ and $U^\dagger = B^\dagger \geqslant 0$. By Theorem 3.3 it then follows that $\rho(B^\dagger(B-A)) < 1$.
(c) ⇒ (a): Let $C \in J$ with $N(C) = N(B)$ and $R(C) = R(B)$. Set $U = B$ and $V = B - C$. Then $C = U - V$, with $R(U) = R(B) = R(C)$ and $N(U) = N(B) = N(C)$. Thus $C = U - V$ is a proper splitting. Also, $0 \leqslant B^\dagger(B-C) \leqslant B^\dagger(B-A)$, so that $\rho(B^\dagger(B-C)) \leqslant \rho(B^\dagger(B-A)) < 1$. It now follows that $C^\dagger \geqslant 0$, by Theorem 3.3. □

The next example shows that there are intervals $J = [A, B]$ that are range kernel regular, with the property that there exists $C \in J \setminus K$ such that $C^\dagger \ngeqslant 0$.

Example 3.1. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$. Then $J = [A, B]$ is range kernel regular. Let $C = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Then $C \in J$, but since C is invertible (and A is not invertible) it follows that $C \notin K$. Also, $C^{\dagger} = C^{-1} \ngeqslant 0$.

Under the circumstances of Theorem 3.5, the next result gives a representation for the Moore–Penrose inverse of $C \in K$.

Lemma 3.1. *Let $J = [A, B]$ be range kernel regular. Suppose that $C \in K$, $B^{\dagger} \geqslant 0$ and $\rho(B^{\dagger}(B - A)) < 1$. Then $C^{\dagger} = \sum_{j=0}^{\infty}(B^{\dagger}(B - C))^j B^{\dagger}$.*

Proof. Observe that $B(I - B^{\dagger}(B - C)) = B - BB^{\dagger}B + BB^{\dagger}C = C$, since $BB^{\dagger} = CC^{\dagger}$ as $0 \leqslant B - C \leqslant B - A$, we also have $0 \leqslant B^{\dagger}(B - C) \leqslant B^{\dagger}(B - A)$. Hence $\rho(B^{\dagger}(B - C)) \leqslant \rho(B^{\dagger}(B - A)) < 1$, so that $I - B^{\dagger}(B - C)$ is invertible. In that case, $(I - B^{\dagger}(B - C))^{-1} = \sum_{j=0}^{\infty}(B^{\dagger}(B - C))^j$. We have $C = B(I - B^{\dagger}(B - C))$. Set $S = I - B^{\dagger}(B - C)$. Then S is invertible. Next, we show that $B^{\dagger}BSS^tB^t = SS^tB^t$. Observe that $BS = C$ so that, $B^{\dagger}BS = B^{\dagger}C$. So, $B^{\dagger}BSS^tB^t = B^{\dagger}CS^tB^t = B^{\dagger}CC^t$. Also $SS^tB^t = SC^t = (I - B^{\dagger}(B - C))C^t = C^t - B^{\dagger}BC^t + B^{\dagger}CC^t = B^{\dagger}CC^t$, where we have used the fact that $B^{\dagger}BC^t = C^{\dagger}CC^t = C^t$. Hence, $B^{\dagger}BSS^tB^t = SS^tB^t$. It now follows from Corollary 2.1, that $C^{\dagger} = (I - B^{\dagger}(B - C))^{-1}B^{\dagger}$. Finally, the representation $C^{\dagger} = \sum_{j=0}^{\infty}(B^{\dagger}(B - C))^j B^{\dagger}$ follows, completing the proof. □

Corollary 3.2. *Let $J = [A, B]$ be range kernel regular. Suppose that $C \in K$ and one of the equivalent conditions (a)–(c) of Theorem 3.5 holds. Then $B^{\dagger} \leqslant C^{\dagger} \leqslant A^{\dagger}$.*

Proof. Since $B^{\dagger} \geqslant 0$ and $C^{\dagger} \geqslant 0$ it follows that $C \leqslant B \Longrightarrow B^{\dagger}CC^{\dagger} \leqslant B^{\dagger}BC^{\dagger} \Longrightarrow B^{\dagger} \leqslant C^{\dagger}$. It can be similarly shown that $C^{\dagger} \leqslant A^{\dagger}$. □

A square matrix A is called a Z-matrix if all the off-diagonal entries of A are nonpositive. A Z-matrix A is called an M-matrix if A can be written as $A = sI - B$, where $s \geqslant \rho(B)$ and $B \geqslant 0$. If $s > \rho(B)$, then A is invertible and $A^{-1} \geqslant 0$. For the singular case (when $s = \rho(B)$) the following result is quite well known. For M-matrices, [5] is an excellent source.

Theorem 3.6 [2, Corollary 5]. *If $A = \rho(B)I - B$, where B is nonnegative and irreducible, then $A^{\dagger} \ngeqslant 0$.*

We also need the following result.

Theorem 3.7 [2, Lemma 4.1]. *Let $A \in \mathbb{R}^{n \times n}$ be a Z-matrix. Then A is an M-matrix if and only if $A + \epsilon I$ is a nonsingular M-matrix for all $\epsilon > 0$.*

Corollary 3.3. *Suppose A and B are singular M-matrices such that $A^{\dagger} \geqslant 0$ and $B^{\dagger} \geqslant 0$. Let J be range kernel regular and $C \in K$. Then C is an M-matrix. Further, if $C = \rho(F)I - F$, where $F \geqslant 0$, then, F is reducible.*

Proof. We have $A \leqslant C \leqslant B$ and so $A + \epsilon I \leqslant C + \epsilon I \leqslant B + \epsilon I$ for each $\epsilon > 0$. Since A and B are singular M-matrices, it follows that $A + \epsilon I$ and $B + \epsilon I$ are invertible M-matrices. Hence $A + \epsilon I$ and $B + \epsilon I$ are inverse positive matrices. So, $C + \epsilon I$ is also inverse positive for each $\epsilon > 0$, by Theorem 1.2. Hence C is an M-matrix, by Theorem 3.7. Also $N(C) = N(B)$ and $R(C) = R(B)$ so that $C^{\dagger} \geqslant 0$, by Theorem 3.5. So, F is reducible, by Theorem 3.6. □

Next, we show that K is nonempty. For a matrix Q, $|\ Q\ |$ denotes the matrix whose components are the absolute values of the corresponding components of Q. In the context of the next result, we recall the result of Beeck (see the references cited in [12]) who has shown that if J_c is invertible and $\rho(|J_c^{-1}|\triangle) < 1$, then each matrix $C \in J$ is invertible, i.e., J is regular.

Theorem 3.8. *Let $J = [A, B]$. Suppose that $N(J_c) = N(A)$, $R(J_c) = R(A)$ and $\rho(|J_c^{\dagger}|\triangle) < 1$. Then K contains the line segment $\lambda A + (1-\lambda)B$, $\lambda \in [0, 1]$. In particular, J is range kernel regular.*

Proof. First, we show that J is range kernel regular. Let $x \in N(A) = N(J_c)$. Then $Ax + Bx = 0$, so that $Bx = 0$. Thus $N(A) \subseteq N(B)$. From the equation $R(J_c) = R(A)$, we have $N(A^t) = N(J_c^t)$ and proceeding as above it follows that $N(A^t) \subseteq N(B^t)$, i.e., $R(B) \subseteq R(A)$. Also, $J_c^{\dagger}(J_c - B) \leqslant |J_c^{\dagger}(J_c - B)| \leqslant |J_c^{\dagger}||J_c - A| = |J_c^{\dagger}|\triangle$. It then follows that $\rho(|\, J_c^{\dagger}(J_c - B)\ |) \leqslant \rho(|\ J_c^{\dagger}\ |\ \triangle) < 1$. Thus the series $\sum_{j=0}^{\infty}(J_c^{\dagger}(J_c - B))^j$ is absolutely convergent and so $\rho(J_c^{\dagger}(J_c - B)) < 1$. Thus $I - J_c^{\dagger}(J_c - B)$ is invertible. Now, $I - (J_c^{\dagger}J_c - J_c^{\dagger}B) = I - P_{R(A^t)} + J_c^{\dagger}B = P_{N(A)} + J_c^{\dagger}B$. Let $D = P_{N(A)} + J_c^{\dagger}B$. Then, $J_cD = J_c(P_{N(A)} + J_c^{\dagger}B) = J_cJ_c^{\dagger}B = B$ (the validity of the last equation is justified by the fact that $J_cJ_c^{\dagger} = P_{R(J_c)} = P_{R(A)}$ and $R(B) \subseteq R(A)$). So, $J_c = BD^{-1}$ and hence $R(J_c) = R(B)$. Thus $R(B) = R(A)$. By the rank-nullity-dimension theorem, it follows that $N(B) = N(A)$. This proves that J is range kernel regular.

Next, set $C = \lambda A + (1-\lambda)B$. We show that $C \in K$. First we prove that $R(C) = R(A)$. Let $x \in R(C)$; $x = Cy = (\lambda A + (1-\lambda)B)y$. Since $R(B) = R(A)$, it then follows that $x \in R(A)$, so that $R(C) \subseteq R(A)$. As argued above, it can be shown that $\rho(J_c^{\dagger}(J_c - C)) < 1$. Again, as before, it follows that $J_c = CD^{-1}$ and hence $R(J_c) = R(C)$. Thus $R(C) = R(A)$. Now, let $x \in N(A)$. Then $x \in N(B)$ and so $x \in N(C)$ so that, $N(A) \subseteq N(C)$. Once again, by the rank-nullity-dimension theorem it follows that $N(C) = N(A)$. □

Remark 3.3. In general, $K \neq J$, even under the hypotheses of Theorem 3.8. This is shown by the following example. We observe that $[A, \alpha A]$ for $A \geqslant 0$, $\alpha > 1$ is an example of a range kernel regular interval matrix. Given $A \in \mathbb{R}^{m\times n}$, $A \geqslant 0$ with at least two nonzero rows, set $B = \alpha A$, $\alpha > 1$. Then $J = [A, B]$ is trivially range kernel regular. Suppose that the kth row of A is nonzero. Define C to be the matrix all of whose entries are the same as that of A except the kth row. Let the *kth* row of C be the kth row of B. Then $C \in J$ and $R(C) \neq R(A)$. For example let $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$. Then J is range kernel regular, $N(J_c) = N(A)$, $R(J_c) = R(A)$, $\rho(|J_c^{\dagger}|\triangle) < 1$, $C \in J$ and $R(C) \neq R(A)$.

Before we conclude this article, we provide another set of sufficient conditions for the range kernel regularity of a bilateral interval.

For $A \in \mathbb{R}^{m\times n}$, a factorization $A = FG$ such that $F \in \mathbb{R}^{m\times r}$, $G \in \mathbb{R}^{r\times n}$ and $r = rank(A) = rank(F) = rank(G)$ is called a full-rank factorization of A. If F and G are (entrywise) nonnegative (and hence A is nonnegative), then such a factorization is called a nonnegative full-rank factorization. There is a well-known result that if $A \geqslant 0$ and there exists $X \geqslant 0$ such that $AXA = A$, then A has a nonnegative full-rank factorization [4].

The next result provides a proper splitting of a matrix A if a full-rank factorization of A is known.

Theorem 3.9 [7, Theorem 3.3]. *Let $A \in \mathbb{R}^{m\times n}$ and $A = FG$ be a full-rank factorization. Then the splitting $A = U - V$ is proper if and only if $U = FSG$ (and $V = U - A$) for some nonsingular $S \in \mathbb{R}^{r\times r}$.*

Let $A \geqslant 0$ and $A^{\dagger} \geqslant 0$. Then (from the comments made as above), it follows that A has a nonnegative full-rank factorization $A = FG$. Suppose that $S \in \mathbb{R}^{r\times r}$, (where r is the rank of A) is invertible and satisfies $S \geqslant I$. Set $B = FSG$. Then $B \geqslant A$. From the Theorem cited above, it then follows that $R(A) = R(B)$ and $N(A) = N(B)$, i.e., the bilateral interval $J = [A, B]$ is range-kernel regular. Let us justify reasonably the assumptions on A and B made as above. In practical applications, the coefficient matrix A is nonnegative. One of the main concerns in this paper is nonnegativity of the Moore–Penrose inverse of the bilateral interval $J = [A, B]$ and one of the assumptions is $A^{\dagger} \geqslant 0$. In practice, B could be thought of as a perturbation of A and hence the assumption that $B = FSG$ with S satisfying $S \geqslant I$ is reasonable. We are only requiring, in addition, that S is invertible.

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