Journal of Functional Analysis 267 (2014) 919–962

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Integral operators on the Oshima compactification of a Riemannian symmetric space ☆

Aprameyan Parthasarathy [a], Pablo Ramacher [b,*]

[a] Institut für Mathematik, Universität Paderborn, Warburger Strasse 100, 33098 Paderborn, Germany
[b] Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, Hans-Meerwein-Str., 35032 Marburg, Germany

A R T I C L E I N F O

Article history:
Received 1 May 2013
Accepted 3 May 2014
Available online 2 June 2014
Communicated by P. Delorme

MSC:
22E46
53C35
32J05
58J40
58J37
58J35
47A10
58C30
20C15

Keywords:
Riemannian symmetric spaces
Compactifications
Kernels
Regularized traces

A B S T R A C T

Consider a Riemannian symmetric space $\mathbb{X} = G/K$ of non-compact type, where G is a connected, real, semisimple Lie group, and K a maximal compact subgroup. Let $\widetilde{\mathbb{X}}$ be its Oshima compactification, and $(\pi, \mathrm{C}(\widetilde{\mathbb{X}}))$ the left-regular representation of G on $\widetilde{\mathbb{X}}$. In this paper, we examine the convolution operators $\pi(f)$ for rapidly decaying functions f on G, and characterize them within the framework of totally characteristic pseudodifferential operators, describing the singular nature of their kernels. As a consequence, we obtain asymptotics for heat and resolvent kernels associated to strongly elliptic operators on $\widetilde{\mathbb{X}}$. As a further application, a regularized trace for the operators $\pi(f)$ can be defined, yielding a distribution on G which can be interpreted as a global character of π, and is given by a fixed point formula analogous to the Atiyah–Bott character formula for an induced representation of G.

☆ The authors wish to thank Toshio Oshima for valuable conversations on the subject. We are also grateful to Jean-Philippe Anker for helpful remarks on the classical heat kernel on Riemannian symmetric spaces. This work was financed by the DFG-grant RA 1370/2-1.

* Corresponding author.
E-mail addresses: apram@math.upb.de (A. Parthasarathy), ramacher@mathematik.uni-marburg.de (P. Ramacher).

http://dx.doi.org/10.1016/j.jfa.2014.05.002
0022-1236/© 2014 Published by Elsevier Inc.

1. Introduction

Let $\mathbb{X}$ be a Riemannian symmetric space of non-compact type. Then $\mathbb{X}$ is isomorphic to G/K, where G is a connected, real, semisimple Lie group, and K a maximal compact subgroup. Consider further the Oshima compactification $\widetilde{\mathbb{X}}$ of $\mathbb{X}$ [16], which is a simply connected, closed, real-analytic manifold carrying an analytic G-action. The orbital decomposition of $\widetilde{\mathbb{X}}$ is of normal crossing type, and the open orbits are isomorphic to G/K, the number of them being equal to 2^l, where l denotes the rank of G/K. In this paper, we study integral operators of the form

$$\pi(f) = \int_G f(g)\pi(g)d_G(g), \tag{1}$$

where π is the regular representation of G on the Banach space $\mathrm{C}(\widetilde{\mathbb{X}})$ of continuous functions on $\widetilde{\mathbb{X}}$, f a smooth, rapidly decreasing function on G, and d_G a Haar measure on G. These operators play an important role in representation theory, and our interest will be directed towards the elucidation of the microlocal structure of the operators $\pi(f)$. Since the underlying group action on $\widetilde{\mathbb{X}}$ is not transitive, the operators $\pi(f)$ are not smooth, and the orbit structure of $\widetilde{\mathbb{X}}$ is reflected in the singular behavior of their Schwartz kernels. As it turns out, the operators in question can be characterized as totally characteristic pseudodifferential operators, a class which was first introduced in [15] in connection with boundary problems. In fact, if $\widetilde{\mathbb{X}}_\Delta$ denotes a component in $\widetilde{\mathbb{X}}$ isomorphic to G/K, we prove that the restrictions

$$\pi(f)_{|\overline{\widetilde{\mathbb{X}}_\Delta}} : \mathrm{C}^\infty_{\mathrm{c}}(\overline{\widetilde{\mathbb{X}}_\Delta}) \to \mathrm{C}^\infty(\overline{\widetilde{\mathbb{X}}_\Delta})$$

of the operators $\pi(f)$ to the manifold with corners $\overline{\widetilde{\mathbb{X}}_\Delta}$ are totally characteristic pseudodifferential operators of class $\mathrm{L}^{-\infty}_b$. A similar structure theorem was already obtained in [18] for integral operators on prehomogeneous vector spaces, but only away from the set of singular points of the complement of the open orbit. In the present case, we are able to achieve a complete description of the operators $\pi(f)$ on $\overline{\widetilde{\mathbb{X}}_\Delta}$ even near the corners due to the fact that the orbital decomposition of $\widetilde{\mathbb{X}}$ is of normal crossing type.

As a first application, we employ the structure theorem to examine the holomorphic semigroup generated by a strongly elliptic operator Ω associated to the regular representation $(\pi, \mathrm{C}(\widetilde{\mathbb{X}}))$ of G, as well as its resolvent. Since both the holomorphic semigroup and the resolvent can be characterized as operators of the form (1), they can be studied applying our structure theorem, and relying on the theory of elliptic operators on Lie groups [19] we obtain a description of the asymptotic behavior of the semigroup and resolvent kernels on $\widetilde{\mathbb{X}}_\Delta \simeq \mathbb{X}$ at infinity. In the particular case of the Laplace–Beltrami operator on $\mathbb{X}$, these questions have been studied intensively before. For the classical heat kernel on $\mathbb{X}$, precise upper and lower bounds were obtained in [1] using spherical analysis, under certain restrictions coming from the lack of control of the Trombi–Varadarajan expansion for spherical functions along the walls. Our results are less explicit, but free of any restrictions, and applicable to a large class of operators. A detailed description of the resolvent of the Laplace–Beltrami operator on $\mathbb{X}$ and its analytic continuation was given in [12–14].

As another consequence of the structure theorem, a regularized trace for the operators $\pi(f)$ is defined, yielding a distribution on the group G which can be thought of as the character of the representation $(\pi, C(\widetilde{\mathbb{X}}))$. In his early work on infinite dimensional representations of semi-simple Lie groups, Harish-Chandra [9] realized that the correct generalization of the character of a finite-dimensional representation was a distribution on the group given by the trace of a convolution operator on representation space. This distribution character is given by a locally integrable function which is analytic on the set of regular elements, and satisfies character formulae analogous to the finite dimensional case. Later, Atiyah and Bott [4] gave a similar description of the character of a parabolically induced representation in their work on Lefschetz fixed point formulae for elliptic complexes. More precisely, let H be a closed, co-compact subgroup of G, and ϱ a representation of H on a finite dimensional vector space V. If $T(g) = (\iota_*\varrho)(g)$ is the representation of G induced by ϱ in the space of sections over G/H with values in the homogeneous vector bundle $G \times_H V$, then its distribution character is given by the distribution

$$\Theta_T : \mathrm{C}^\infty_\mathrm{c}(G) \ni f \mapsto \operatorname{Tr} T(f), \quad T(f) = \int_G f(g) T(g) d_G(g),$$

where d_G denotes a Haar measure on G. The point to be noted is that $T(f)$ is a smooth operator, and since G/H is compact, it does have a well-defined trace. On the other hand, assume that $g \in G$ is transversal, meaning that it acts on G/H only with simple fixed points. In this case, a flat trace $\operatorname{Tr}^\flat T(g)$ of $T(g)$ can be defined within the framework of pseudodifferential operators, which is given by a sum over fixed points of g. Atiyah and Bott then showed that, on an open set $G_T \subset G$ of transversal elements,

$$\Theta_T(f) = \int_{G_T} f(g) \operatorname{Tr}^\flat T(g) d_G(g), \quad f \in \mathrm{C}^\infty_\mathrm{c}(G_T).$$

This means that, on G_T, the character Θ_T of the induced representation T is represented by the locally integrable function $\operatorname{Tr}^\flat T(g)$, and its computation reduced to the evaluation of a sum over fixed points. In our case, contrasting with the classical homogeneous setting, the convolution operators $\pi(f)$ are not smooth due to the presence of the lower-dimensional orbits, and therefore do not have a well-defined trace. Nevertheless, by showing that they can be characterized as totally characteristic pseudodifferential operators of order $-\infty$, we are able to define a regularized trace $\operatorname{Tr}_{reg} \pi(f)$ for the operators $\pi(f)$, and in this way obtain a map

$$\Theta_\pi : \mathrm{C}^\infty_\mathrm{c}(G) \ni f \mapsto \mathrm{Tr}_{reg}(f) \in \mathbb{C},$$

which is shown to be a distribution on G. This distribution can be thought of as the character of the representation π. We then show that, on a certain open set $G(\widetilde{\mathbb{X}})$ of transversal elements,

$$\mathrm{Tr}_{reg}\, \pi(f) = \int\limits_{G(\widetilde{\mathbb{X}})} f(g)\, \mathrm{Tr}^\flat\, \pi(g) d_G(g), \quad f \in \mathrm{C}^\infty_\mathrm{c}\big(G(\widetilde{\mathbb{X}})\big),$$

where, with the notation $\Phi_g(\tilde{x}) = g^{-1} \cdot \tilde{x}$,

$$\mathrm{Tr}^\flat\, \pi(g) = \sum_{\tilde{x} \in \mathrm{Fix}(\widetilde{\mathbb{X}}, g)} \frac{1}{|\det(\mathbf{1} - d\Phi_g(\tilde{x}))|},$$

the sum being over the (simple) fixed points of $g \in G(\widetilde{\mathbb{X}})$ on $\widetilde{\mathbb{X}}$. Thus, on the open set $G(\widetilde{\mathbb{X}})$, Θ_π is represented by the locally integrable function $\mathrm{Tr}^\flat\, \pi(g)$, which is given by a formula similar to the character of a parabolically induced representation. In a subsequent work, the authors intend to interpret Θ_π in representation theoretic terms, and to describe the singularities of Θ_π in a more detailed way. Furthermore, it is natural to ask whether similar distribution characters can be introduced on spherical varieties, which are normal algebraic varieties with the action of a reductive algebraic group, and a Zariski-dense orbit of a Borel subgroup, and whether corresponding character formulae can be proved. Such characters are expected to be relevant in the context of harmonic analysis on spherical varieties.

The paper is organized as follows. In Section 2 we recall those parts of the structure theory of real, semisimple Lie groups that are relevant to our purposes. We then describe the G-action on certain homogeneous spaces $G/P_\Theta(K)$, where $P_\Theta(K)$ is a closed subgroup of G associated naturally to a subset Θ of the set of simple roots, and the corresponding fundamental vector fields. This leads to the definition of the Oshima compactification $\widetilde{\mathbb{X}}$ of the symmetric space $\mathbb{X} = G/K$, together with a description of the orbital decomposition of $\widetilde{\mathbb{X}}$. Since this decomposition is of normal crossing type, it is well-suited for our analytic purposes. A thorough and unified description of the various compactifications of a symmetric space is given in [6]. Section 3 contains a summary of some basic facts in the theory pseudodifferential operators needed in the sequel. In particular, the class of totally characteristic pseudodifferential operators on a manifold with corners is introduced. Section 4 is the central part of this paper. By analyzing the orbit structure of the G-action on $\widetilde{\mathbb{X}}$, we are able to elucidate the microlocal structure of the convolution operators $\pi(f)$, and characterize them as totally characteristic pseudodifferential operators on the manifold with corners $\overline{\widetilde{\mathbb{X}}_\Delta}$. This leads to a description of the asymptotic behavior of their Schwartz kernels at infinity when approaching the boundary of $\widetilde{\mathbb{X}}_\Delta \simeq \mathbb{X}$. In Section 5, we consider the holomorphic semigroup S_τ generated by the closure $\overline{\Omega}$ of a strongly elliptic differential operator Ω associated to the representation π.

Since $S_\tau = \pi(f_\tau)$, where $f_\tau(g)$ is a smooth and rapidly decreasing function on G, we can apply our previous results to describe the Schwartz kernel of S_τ. The treatment of the Schwartz kernel of the resolvent $(\lambda \mathbf{1} + \overline{\Omega})^{-\alpha}$, where $\alpha > 0$, and $\operatorname{Re}\lambda$ is sufficiently large, is similar, but subtler due to the singularity of the corresponding group kernel $r_{\alpha,\lambda}(g)$ at the identity. The regularized trace for the convolution operators $\pi(f)$ is defined in Section 6. After studying fixed points of G-actions on homogeneous spaces in Section 7, and introducing the transversal trace of a pseudodifferential operator in Section 8, we finally prove that the distribution Θ_π is regular on the set of transversal elements $G(\widetilde{\mathbb{X}})$, and given by the locally integrable function $\operatorname{Tr}^\flat \pi(g)$.

2. The Oshima compactification of a Riemannian symmetric space

Let G be a connected, real, semisimple Lie group with finite center and Lie algebra $\mathfrak{g}$, and denote by $\langle X, Y\rangle = \operatorname{tr}(\operatorname{ad} X \circ \operatorname{ad} Y)$ the *Cartan-Killing form* on $\mathfrak{g}$. Let θ be a Cartan involution of $\mathfrak{g}$, and let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be the Cartan decomposition of $\mathfrak{g}$ into the eigenspaces of θ, corresponding to the eigenvalues $+1$ and -1, respectively, and put $\langle X, Y\rangle_\theta := -\langle X, \theta Y\rangle$. Note that the Cartan decomposition is orthogonal with respect to $\langle\,,\rangle_\theta$. Consider further a maximal Abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. The dimension l of $\mathfrak{a}$ is called the *real rank of* G and *the rank of the symmetric space* G/K. Then $\operatorname{ad}(\mathfrak{a})$ is a commuting family of self-adjoint operators on $\mathfrak{g}$. Next, one defines for each $\alpha \in \mathfrak{a}^*$, the dual of $\mathfrak{a}$, the simultaneous eigenspaces $\mathfrak{g}^\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$ of $\operatorname{ad}(\mathfrak{a})$. A functional $0 \neq \alpha \in \mathfrak{a}^*$ is called a *(restricted) root* of $(\mathfrak{g}, \mathfrak{a})$ if $\mathfrak{g}^\alpha \neq \{0\}$, and setting $\Sigma = \{\alpha \in \mathfrak{a}^* : \alpha \neq 0,\ \mathfrak{g}^\alpha \neq \{0\}\}$, we obtain the decomposition

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}^\alpha,$$

where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Note that this decomposition is orthogonal with respect to $\langle\cdot,\cdot\rangle_\theta$. With respect to an ordering of $\mathfrak{a}^*$, let $\Sigma^+ = \{\alpha \in \Sigma : \alpha > 0\}$ denote the *set of positive roots*, and $\Delta = \{\alpha_1, \dots, \alpha_l\}$ the *set of simple roots*. Let $\varrho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} \alpha$, and put $m(\alpha) = \dim \mathfrak{g}^\alpha$ which is, in general, greater than 1. Define $\mathfrak{n}^+ = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}^\alpha$, $\mathfrak{n}^- = \theta(\mathfrak{n}^+)$, and write K, A, N^+ and N^- for the analytic subgroups of G corresponding to $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}^+$, and $\mathfrak{n}^-$, respectively. The *Iwasawa decomposition* of G is then given by

$$G = KAN^\pm.$$

Next, let $M = \{k \in K : \operatorname{Ad}(k)H = H \text{ for all } H \in \mathfrak{a}\}$ be the centralizer of $\mathfrak{a}$ in K and $M^* = \{k \in K : \operatorname{Ad}(k)\mathfrak{a} \subset \mathfrak{a}\}$ the normalizer of $\mathfrak{a}$ in K. The quotient $W = M^*/M$ is the *Weyl group* corresponding to $(\mathfrak{g}, \mathfrak{a})$, and acts on $\mathfrak{a}$ as a group of linear transformations via the adjoint action. Alternatively, W can be characterized as follows. For each $\alpha_i \in \Delta$, define a reflection in $\mathfrak{a}^*$ with respect to the Cartan-Killing form $\langle\cdot,\cdot\rangle$ by

$$w_{\alpha_i} : \lambda \mapsto \lambda - 2\alpha_i \langle \lambda, \alpha_i\rangle / \langle \alpha_i, \alpha_i\rangle,$$

where $\langle \lambda, \alpha\rangle = \langle H_\lambda, H_\alpha\rangle$. Here H_λ is the unique element in $\mathfrak{a}$ corresponding to a given $\lambda \in \mathfrak{a}^*$, and is determined by the non-degeneracy of the Cartan-Killing form. One can then identify the Weyl group W with the group generated by the reflections $\{w_{\alpha_i} : \alpha_i \in \Delta\}$. For a subset Θ of Δ, let W_Θ denote the subgroup of W generated by reflections corresponding to elements in Θ, and define

$$P_\Theta = \bigcup_{w \in W_\Theta} P m_w P,$$

where m_w denotes a representative of w in M^*, and $P = MAN^+$ is a minimal parabolic subgroup. It is then a classical result in the theory of parabolic subgroups [22] that, as Θ ranges over the subsets of Δ, one obtains, in this way, all the parabolic subgroups of G containing P. In particular, if $\Theta = \emptyset$, $P_\Theta = P$. Let us now introduce for $\Theta \subset \Delta$ the subalgebras

$$\begin{aligned} \mathfrak{a}_\Theta &= \bigl\{ H \in \mathfrak{a} : \alpha(H) = 0 \text{ for all } \alpha \in \Theta \bigr\}, \\ \mathfrak{a}(\Theta) &= \bigl\{ H \in \mathfrak{a} : \langle H, X\rangle_\theta = 0 \text{ for all } X \in \mathfrak{a}_\Theta \bigr\}. \end{aligned}$$

Note that, when restricted to the $+1$ or the -1 eigenspace of θ, the orthogonal complement of a subspace with respect to $\langle\cdot,\cdot\rangle$ is the same as its orthogonal complement with respect to $\langle\cdot,\cdot\rangle_\theta$. We further define

$$\begin{aligned} \mathfrak{n}^+_\Theta &= \sum_{\alpha \in \Sigma^+ \setminus \langle\Theta\rangle^+} \mathfrak{g}^\alpha, & \mathfrak{n}^-_\Theta &= \theta\bigl(\mathfrak{n}^+_\Theta\bigr), \\ \mathfrak{n}^+(\Theta) &= \sum_{\alpha \in \langle\Theta\rangle^+} \mathfrak{g}^\alpha, & \mathfrak{n}^-(\Theta) &= \theta\bigl(\mathfrak{n}^+(\Theta)\bigr), \\ \mathfrak{m}_\Theta &= \mathfrak{m} + \mathfrak{n}^+(\Theta) + \mathfrak{n}^-(\Theta) + \mathfrak{a}(\Theta), & \mathfrak{m}_\Theta(K) &= \mathfrak{m}_\Theta \cap \mathfrak{k}, \end{aligned}$$

where $\langle\Theta\rangle^+ = \Sigma^+ \cap \sum_{\alpha_i \in \Theta} \mathbb{R}\alpha_i$. Denoting by A_Θ, $A(\Theta)$, $N^\pm_\Theta$, $N^\pm(\Theta)$, $M_{\Theta,0}$, and $M_\Theta(K)_0$, the corresponding connected analytic subgroups of G, we obtain the decompositions $A = A_\Theta A(\Theta)$ and $N^\pm = N^\pm_\Theta N(\Theta)^\pm$, the second being a semi-direct product. Let next $M_\Theta = M M_{\Theta,0}$, $M_\Theta(K) = M M_\Theta(K)_0$. One has the *Iwasawa decompositions*

$$M_\Theta = M_\Theta(K) A(\Theta) N^\pm(\Theta),$$

and the *Langlands decompositions*

$$P_\Theta = M_\Theta A_\Theta N^+_\Theta = M_\Theta(K) A N^+.$$

In particular, $P_\Delta = M_\Delta = G$, since $m_\Delta = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}^\alpha$, and $\mathfrak{a}_\Delta, \mathfrak{n}^+_\Delta$ are trivial. One then defines

$$P_\Theta(K) = M_\Theta(K) A_\Theta N^+_\Theta \subset P_\Theta.$$

According to [16, Lemma 1], $P_\Theta(K)$ is a closed subgroup, and G is a union of the open and dense submanifold $N^- A(\Theta) P_\Theta(K) = N^-_\Theta P_\Theta$, and submanifolds of lower dimension. For $\Delta = \{\alpha_1, \dots, \alpha_l\}$, let next $\{H_1, \dots, H_l\}$ be the basis of $\mathfrak{a}$, dual to Δ, i.e. $\alpha_i(H_j) = \delta_{ij}$. Fix a basis $\{X_{\lambda,i} : 1 \leqslant i \leqslant m(\lambda)\}$ of $\mathfrak{g}^\lambda$ for each $\lambda \in \Sigma^+$. Clearly,

$$[H, -\theta X_{\lambda,i}] = -\theta[\theta H, X_{\lambda,i}] = -\lambda(H)(-\theta X_{\lambda,i}), \quad H \in \mathfrak{a},$$

so that setting $X_{-\lambda,i} = -\theta(X_{\lambda,i})$, one obtains a basis $\{X_{-\lambda,i} : 1 \leqslant i \leqslant m(\lambda)\}$ of $\mathfrak{g}^{-\lambda} \subset \mathfrak{n}^-$. One now has the following lemma, due to Oshima, which gives a description of the infinitesimal action of G.

Lemma 1. *Fix an element $g \in G$, and identify $N^- \times A(\Theta)$ with an open dense submanifold of the homogeneous space $G/P_\Theta(K)$ by the map $(n, a) \mapsto gnaP_\Theta(K)$. For $Y \in \mathfrak{g}$, let $Y_{|G/P_\Theta(K)}$ be the fundamental vector field corresponding to the action of the one-parameter group $\exp(sY)$, $s \in \mathbb{R}$, on $G/P_\Theta(K)$. Then, at any point $p = (n, a) \in N^- \times A(\Theta)$, we have*

$$(Y_{|G/P_\Theta(K)})_p = \sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} c_{-\lambda,i}(g, n)(X_{-\lambda,i})_p + \sum_{\lambda \in \langle\Theta\rangle^+} \sum_{i=1}^{m(\lambda)} c_{\lambda,i}(g, n) e^{-2\lambda(\log a)}(X_{-\lambda,i})_p$$
$$+ \sum_{\alpha_i \in \Theta} c_i(g, n)(H_i)_p$$

with the identification $T_n N^- \oplus T_a(A(\Theta)) \simeq T_p(N^- \times A(\Theta)) \simeq T_{gnaP_\Theta(K)} G/P_\Theta(K)$. The coefficient functions $c_{\lambda,i}(g, n)$, $c_{-\lambda,i}(g, n)$, $c_i(g, n)$ are real-analytic, and are determined by the equation

$$\mathrm{Ad}^{-1}(gn)Y = \sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} \bigl(c_{\lambda,i}(g, n) X_{\lambda,i} + c_{-\lambda,i}(g, n) X_{-\lambda,i}\bigr)$$
$$+ \sum_{i=1}^{l} c_i(g, n) H_i \bmod \mathfrak{m}. \tag{2}$$

Proof. For a detailed proof following the original proof given in [16, Lemma 3], we refer to [17]. □

By the identification $G/K \simeq N^- \times A \simeq N^- \times \mathbb{R}^l_+$ via $(n, t) \mapsto n \cdot \exp(-\sum_{i=1}^l H_i \times \log t_i) \mapsto gnaK$ one sees that

$$H_{i|N^- \times \mathbb{R}^l_+} = -t_i \frac{\partial}{\partial t_i}.$$

Therefore, the action on G/K of the fundamental vector field corresponding to $\exp(sY)$, $Y \in \mathfrak{g}$, is given by

$$Y_{|N^- \times \mathbb{R}^l_+} = \sum_{\lambda \in \Sigma^+} \sum_{i=1}^{m(\lambda)} \big(c_{\lambda,i}(g,n)t^{2\lambda} + c_{-\lambda,i}(g,n)\big) X_{-\lambda,i} - \sum_{i=1}^{l} c_i(g,n) t_i \frac{\partial}{\partial t_i}, \tag{3}$$

where the coefficients are given by (2), and where we wrote $t^\lambda = t_1^{\lambda(H_1)} \cdots t_l^{\lambda(H_l)}$. The vector field (3) can be extended analytically to $N^- \times \mathbb{R}^l$ as there are no negative powers of t.

We come now to the description of the Oshima compactification of the Riemannian symmetric space G/K. For this, let $\hat{\mathbb{X}}$ be the product manifold $G \times N^- \times \mathbb{R}^l$. Take $\hat{x} = (g,n,t) \in \hat{\mathbb{X}}$, where $g \in G$, $n \in N^-$, $t = (t_1, \dots, t_l) \in \mathbb{R}^l$, and define an action of G on $\hat{\mathbb{X}}$ by $g' \cdot (g,n,t) := (g'g,n,t)$, $g' \in G$. For $s \in \mathbb{R}$, let

$$\operatorname{sgn} s = \begin{cases} s/|s|, & s \neq 0, \\ 0, & s = 0, \end{cases}$$

and put $\operatorname{sgn} \hat{x} = (\operatorname{sgn} t_1, \dots, \operatorname{sgn} t_l) \in \{-1,0,1\}^l$. We then define the subsets $\Theta_{\hat{x}} = \{\alpha_i \in \Delta : t_i \neq 0\}$. Similarly, let $a(\hat{x}) = \exp(-\sum_{t_i \neq 0} H_i \log|t_i|) \in A(\Theta_{\hat{x}})$. On $\hat{\mathbb{X}}$, define now an equivalence relation by setting

$$\hat{x} = (g,n,t) \sim \hat{x}' = \big(g', n,' t'\big) \quad \Leftrightarrow \quad \begin{cases} (a) \ \operatorname{sgn} \hat{x} = \operatorname{sgn} \hat{x}', \\ (b) \ gna(\hat{x}) P_{\Theta_{\hat{x}}}(K) = g'n'a(\hat{x}') P_{\Theta_{\hat{x}'}}(K). \end{cases}$$

Note that the condition $\operatorname{sgn} \hat{x} = \operatorname{sgn} \hat{x}'$ implies that $\hat{x}, \hat{x}'$ determine the same subset $\Theta_{\hat{x}}$ of Δ, and consequently the same group $P_{\Theta_{\hat{x}}}(K)$, as well as the same homogeneous space $G/P_{\Theta_{\hat{x}}}(K)$, so that condition (b) makes sense. It says that $gna(\hat{x}), g'n'a(\hat{x}')$ are in the same $P_{\Theta_{\hat{x}}}(K)$ orbit on G, corresponding to the right action by $P_{\Theta_{\hat{x}}}(K)$ on G. We now define

$$\widetilde{\mathbb{X}} := \hat{\mathbb{X}} / \sim,$$

endowing it with the quotient topology, and denote by $\pi : \hat{\mathbb{X}} \to \widetilde{\mathbb{X}}$ the canonical projection. The action of G on $\hat{\mathbb{X}}$ is compatible with the equivalence relation $\sim$, yielding a G-action $g' \cdot \pi(g,n,t) := \pi(g'g,n,t)$ on $\widetilde{\mathbb{X}}$. For each $g \in G$, one can show that the maps

$$\varphi_g : N^- \times \mathbb{R}^l \to \widetilde{U}_g : (n,t) \mapsto \pi(g,n,t), \quad \widetilde{U}_g = \pi\big(\{g\} \times N^- \times \mathbb{R}^l\big),$$

are bijections. One has then the following

Theorem 1.

(1) $\widetilde{\mathbb{X}}$ *is a simply connected, compact, real-analytic manifold without boundary.*

(2) $\widetilde{\mathbb{X}} = \bigcup_{w \in W} \widetilde{U}_{m_w} = \bigcup_{g \in G} \widetilde{U}_g$. *For* $g \in G$, $\widetilde{U}_g$ *is an open submanifold of* $\widetilde{\mathbb{X}}$ *topologized in such a way that the coordinate map* φ_g *defined above is a real-analytic diffeomorphism. Furthermore,* $\widetilde{\mathbb{X}} \setminus \widetilde{U}_g$ *is the union of a finite number of submanifolds of* $\widetilde{\mathbb{X}}$ *whose codimensions in* $\widetilde{\mathbb{X}}$ *are not lower than* 2.

(3) *The action of* G *on* $\widetilde{\mathbb{X}}$ *is real-analytic. For a point* $\hat{x} \in \hat{\mathbb{X}}$, *the* G*-orbit of* $\pi(\hat{x})$ *is isomorphic to the homogeneous space* $G/P_{\Theta_{\hat{x}}}(K)$, *and for* $\hat{x}, \hat{x}' \in \hat{\mathbb{X}}$ *the* G*-orbits of* $\pi(\hat{x})$ *and* $\pi(\hat{x}')$ *coincide if and only if* $\operatorname{sgn} \hat{x} = \operatorname{sgn} \hat{x}'$. *Hence the orbital decomposition of* $\widetilde{\mathbb{X}}$ *with respect to the action of* G *is of the form*

$$\widetilde{\mathbb{X}} \simeq \bigsqcup_{\Theta \subset \Delta} 2^{\#\Theta} \big(G/P_\Theta(K)\big) \quad (\textit{disjoint union}), \tag{4}$$

where $\#\Theta$ *is the number of elements of* Θ *and* $2^{\#\Theta}(G/P_\Theta(K))$ *is the disjoint union of* $2^{\#\Theta}$ *copies of* $G/P_\Theta(K)$.

Proof. See Oshima, [16, Theorem 5]. □

Observe that the theorem tells us, in particular, that there are 2^l open orbits all of which are isomorphic to G/K, and a unique closed orbit isomorphic to G/P. Next, define for $\hat{x} = (g, n, t)$ the set $B_{\hat{x}} = \{(t'_1 \dots t'_l) \in \mathbb{R}^l : \operatorname{sgn} t_i = \operatorname{sgn} t'_i,\ 1 \leqslant i \leqslant l\}$. By analytic continuation, one can restrict the vector field (3) to $N^- \times B_{\hat{x}}$, and with the identifications $G/P_{\Theta_{\hat{x}}} \simeq N^- \times A(\Theta_{\hat{x}}) \simeq N^- \times B_{\hat{x}}$ via the maps

$$gnaP_{\Theta_{\hat{x}}} \leftarrow (n, a) \mapsto \big(n, \operatorname{sgn} t_1 e^{-\alpha_1(\log a)}, \dots, \operatorname{sgn} t_l e^{-\alpha_l(\log a)}\big),$$

one actually sees that this restriction coincides with the vector field in Lemma 1. The action of the fundamental vector field on $\widetilde{\mathbb{X}}$ corresponding to $\exp sY$, $Y \in \mathfrak{g}$, is therefore given by the extension of (3) to $N^- \times \mathbb{R}^l$. Note that for a simply connected nilpotent Lie group N with Lie algebra $\mathfrak{n}$, the exponential $\exp : \mathfrak{n} \to N$ is a diffeomorphism. So, in our setting, we can identify N^- with $\mathbb{R}^k$. Thus, for every point in $\widetilde{\mathbb{X}}$, there exists a local coordinate system $(n_1, \dots, n_k, t_1, \dots, t_l)$ in a neighborhood of that point such that two points $(n_1, \dots, n_k, t_1, \dots, t_l)$ and $(n'_1, \dots, n'_k, t'_1, \dots, t'_l)$ belong to the same G-orbit if, and only if, $\operatorname{sgn} t_j = \operatorname{sgn} t'_j$, for $j = 1, \dots, l$. This means that the orbital decomposition of $\widetilde{\mathbb{X}}$ is of *normal crossing type*. In what follows, we shall identify the open G-orbit $\pi(\{\hat{x} = (e, n, t) \in \hat{\mathbb{X}} : \operatorname{sgn} \hat{x} = (1, \dots, 1)\})$ with the Riemannian symmetric space G/K, and the orbit $\pi(\{\hat{x} \in \hat{\mathbb{X}} : \operatorname{sgn} \hat{x} = (0, \dots, 0)\})$ of lowest dimension with its Martin boundary G/P.

3. Review of pseudodifferential operators

3.1. Generalities

In this section, we shall briefly recall some basic facts about pseudodifferential operators needed to formulate our main results in the sequel. For a detailed exposition, the reader is referred to [10] and [20]. Let U be an open set in $\mathbb{R}^n$. A continuous linear operator

$$A : \mathrm{C}_{\mathrm{c}}^{\infty}(U) \to \mathrm{C}^{\infty}(U)$$

is called a *pseudodifferential operator of order* $l \in \mathbb{R}$ if it is of the form

$$Au(x) = \int e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)\,\bar{d}\xi, \tag{5}$$

where $\hat{u}$ denotes the Fourier transform of u, $\bar{d}\xi = (2\pi)^{-n} d\xi$, and the amplitude a belongs to the symbol class $\mathrm{S}^l(U \times \mathbb{R}^n)$ of smooth functions satisfying the estimates

$$\left|\left(\partial_\xi^\alpha \partial_x^\beta a\right)(x,\xi)\right| \leqslant C_{\alpha,\beta,\mathcal{K}}\left(1+|\xi|^2\right)^{(l-|\alpha|)/2}, \quad x \in \mathcal{K},\ \xi \in \mathbb{R}^n,$$

for any multi-indices α, β, any compact set $\mathcal{K} \subset U$, and suitable constants $C_{\alpha,\beta,\mathcal{K}} > 0$. The Schwartz kernel of A is a distribution $K_A \in \mathcal{D}'(U \times U)$ given by the oscillatory integral

$$K_A(x,y) = \int e^{i(x-y)\cdot\xi} a(x,\xi)\,\bar{d}\xi,$$

and is a smooth function off the diagonal in $U \times U$. The class of all such operators is denoted by $\mathrm{L}^l(U)$ and the set $\mathrm{L}^{-\infty}(U) = \bigcap_{l\in\mathbb{R}} \mathrm{L}^l(U)$ consists of all operators with smooth kernel, or *smooth operators*. Consider next an n-dimensional paracompact C^∞ manifold $\mathbf{X}$, and let $\{(\kappa_\gamma, \widetilde{U}^\gamma)\}$ be an atlas for $\mathbf{X}$. Then a linear operator

$$A : \mathrm{C}_{\mathrm{c}}^{\infty}(\mathbf{X}) \to \mathrm{C}^{\infty}(\mathbf{X}) \tag{6}$$

is called a *pseudodifferential operator on* $\mathbf{X}$ *of order* l if for each chart diffeomorphism $\kappa_\gamma : \widetilde{U}^\gamma \to U^\gamma = \kappa_\gamma(\widetilde{U}^\gamma)$, the operator $A^\gamma u = [A_{|\widetilde{U}^\gamma}(u \circ \kappa_\gamma)] \circ \kappa_\gamma^{-1}$ given by the diagram

$$\begin{tikzcd}
\mathrm{C}_{\mathrm{c}}^{\infty}(\widetilde{U}^\gamma) \arrow[r, "A_{|\widetilde{U}^\gamma}"] & \mathrm{C}^{\infty}(\widetilde{U}^\gamma) \\
\mathrm{C}_{\mathrm{c}}^{\infty}(U^\gamma) \arrow[r, "A^\gamma"] \arrow[u, "\kappa_\gamma^*"] & \mathrm{C}^{\infty}(U^\gamma) \arrow[u, "\kappa_\gamma^*"']
\end{tikzcd}$$

is a pseudodifferential operator on U^γ of order l, and its Schwartz kernel K_A is smooth off the diagonal. In this case we write $A \in \mathrm{L}^l(\mathbf{X})$. Note that, since the $\widetilde{U}^\gamma$ are not necessarily connected, we can choose them in such a way that $\mathbf{X} \times \mathbf{X}$ is covered by the open sets $\widetilde{U}^\gamma \times \widetilde{U}^\gamma$. Therefore the condition that K_A is smooth off the diagonal can be dropped. The kernel of A is determined by the kernels $K_{A^\gamma} \in \mathcal{D}'(U^\gamma \times U^\gamma)$. If $l < -\dim \mathbf{X}$, they are continuous, and given by absolutely convergent integrals. In this case, their restrictions to the respective diagonals in $U^\gamma \times U^\gamma$ define continuous functions

$$k^\gamma(\tilde{x}) = K_{A^\gamma}\big(\kappa_\gamma(\tilde{x}), \kappa_\gamma(\tilde{x})\big), \quad \tilde{x} \in \widetilde{U}^\gamma,$$

which, for $\tilde{x} \in \widetilde{U}^{\gamma_1} \cap \widetilde{U}^{\gamma_2}$, satisfy the relations $k^{\gamma_2}(\tilde{x}) = |\det(\kappa_{\gamma_1} \circ \kappa_{\gamma_2}^{-1})'| \circ \kappa_{\gamma_2}(\tilde{x}) k^{\gamma_1}(\tilde{x})$, and therefore define a density $k \in C(\mathbf{X}, \Omega)$ on $\Delta_{\mathbf{X}\times\mathbf{X}} \simeq \mathbf{X}$, where Ω denotes the density bundle on $\mathbf{X}$. If $\mathbf{X}$ is compact, this density can be integrated, yielding the trace of the operator A,

$$\operatorname{tr} A = \int_{\mathbf{X}} k = \sum_\gamma \int_{U^\gamma} \big(\alpha_\gamma \circ \kappa_\gamma^{-1}\big)(x) K_{A^\gamma}(x,x)\,dx, \tag{7}$$

where $\{\alpha_\gamma\}$ denotes a partition of unity subordinated to the atlas $\{(\kappa_\gamma, \widetilde{U}^\gamma)\}$, and dx is Lebesgue measure in $\mathbb{R}^n$.

3.2. Totally characteristic pseudodifferential operators

We introduce now a special class of pseudodifferential operators associated in a natural way to a C^∞ manifold $\mathbf{X}$ with boundary $\partial\mathbf{X}$. Our main reference will be [15] in this case. Let $\mathrm{C}^\infty(\mathbf{X})$ be the space of functions on $\mathbf{X}$ which are C^∞ up to the boundary, and $\dot{\mathrm{C}}^\infty(\mathbf{X})$ the subspace of functions vanishing to all orders on $\partial\mathbf{X}$, and define corresponding spaces of distributions over $\mathbf{X}$ by

$$\mathcal{D}'(\mathbf{X}) = \big(\dot{\mathrm{C}}^\infty_{\mathrm{c}}(\mathbf{X}, \Omega)\big)', \qquad \dot{\mathcal{D}}(\mathbf{X})' = \big(\mathrm{C}^\infty_{\mathrm{c}}(\mathbf{X}, \Omega)\big)'.$$

Consider the translated partial Fourier transform of a symbol $a(x,\xi) \in \mathrm{S}^l(\mathbb{R}^n \times \mathbb{R}^n)$,

$$Ma\big(x, \xi'; t\big) = \int e^{i(1-t)\xi_1} a\big(x, \xi_1, \xi'\big)\,d\xi_1,$$

where we wrote $\xi = (\xi_1, \xi')$. $Ma(x, \xi'; t)$ is C^∞ away from $t = 1$, and one says that $a(x,\xi)$ is *lacunary* if it satisfies the condition

$$Ma\big(x, \xi'; t\big) = 0 \quad \text{for } t < 0. \tag{8}$$

The subspace of lacunary symbols will be denoted by $\mathrm{S}^l_{la}(\mathbb{R}^n \times \mathbb{R}^n)$. Let $Z = \overline{\mathbb{R}^+} \times \mathbb{R}^{n-1}$ be the standard manifold with boundary with the natural coordinates $x = (x_1, x')$. In order to define on Z operators of the form (5), where now $a(x,\xi) = \widetilde{a}(x_1, x', x_1\xi_1, \xi')$ is a more general amplitude and $\widetilde{a}(x,\xi)$ is lacunary, one considers the formal adjoint A^* of A and shows that it defines a separately continuous form

$$\mathrm{S}^{\infty}_{la}\big(Z\times\mathbb{R}^n\big)\times \mathrm{C}^{\infty}_{\mathrm{c}}(Z)\to \mathrm{C}^{\infty}(Z),$$

see [15, Propositions 3.6 and 3.9]. For $\widetilde{a}\in \mathrm{S}^{\infty}_{la}(Z\times\mathbb{R}^n)$, one then defines the operator

$$A:\dot{\mathcal{E}}'(Z)\to\dot{\mathcal{D}}'(Z), \tag{9}$$

written formally as (5), as the adjoint of A^*. The space $\mathrm{L}^l_b(Z)$ of *totally characteristic pseudodifferential operators on* Z *of order* l consists of those continuous linear maps (9) such that for any $u,v\in \mathrm{C}^{\infty}_{\mathrm{c}}(Z)$, vAu is of the form (5) with $a(x,\xi)=\widetilde{a}(x_1,x',x_1\xi_1,\xi')$ and $\widetilde{a}(x,\xi)\in \mathrm{S}^l_{la}(Z\times\mathbb{R}^n)$. Similarly, a continuous linear map (6) on a smooth manifold $\mathbf{X}$ with boundary $\partial\mathbf{X}$ is said to be an element of the space $\mathrm{L}^l_b(\mathbf{X})$ of *totally characteristic pseudodifferential operators on* $\mathbf{X}$ *of order* l, if for a given atlas $\{(\kappa_\gamma,\widetilde{U}^\gamma)\}$ the operators $A^\gamma u=[A_{|\widetilde{U}^\gamma}(u\circ\kappa_\gamma)]\circ\kappa_\gamma^{-1}$ are elements of $\mathrm{L}^l_b(Z)$, where the $\widetilde{U}^\gamma$ are coordinate patches isomorphic to subsets in Z.

In an analogous way, it is possible to introduce the concept of a totally characteristic pseudodifferential operator on a manifold with corners. As the standard manifold with corners, consider

$$\mathbb{R}^{n,k}=[0,\infty)^k\times\mathbb{R}^{n-k},\quad 0\leqslant k\leqslant n,$$

with coordinates $x=(x_1,\dots,x_k,x')$. Under a *totally characteristic pseudodifferential operator on* $\mathbb{R}^{n,k}$ *of order* l we shall understand a continuous linear operator which is locally given by an oscillatory integral (5) with $a(x,\xi)=\widetilde{a}(x,x_1\xi_1,\dots,x_k\xi_k,\xi')$, where now $\widetilde{a}(x,\xi)$ is a symbol of order l that satisfies the lacunary condition for each of the coordinates $x_1,\dots,x_k$, i.e.

$$\int e^{i(1-t)\xi_j}a(x,\xi)\,d\xi_j=0\quad\text{for } t<0 \text{ and } 1\leqslant j\leqslant k.$$

In this case, we write $\widetilde{a}(x,\xi)\in \mathrm{S}^l_{la}(\mathbb{R}^{n,k}\times\mathbb{R}^n)$. A continuous linear map (6) on a smooth manifold $\mathbf{X}$ with corners is then said to be an element of the space $\mathrm{L}^l_b(\mathbf{X})$ of *totally characteristic pseudodifferential operators on* $\mathbf{X}$ *of order* l, if for a given atlas $\{(\kappa_\gamma,\widetilde{U}^\gamma)\}$ the operators $A^\gamma u=[A_{|\widetilde{U}^\gamma}(u\circ\kappa_\gamma)]\circ\kappa_\gamma^{-1}$ are totally characteristic pseudodifferential operators on $\mathbb{R}^{n,k}$ of order l, where the $\widetilde{U}^\gamma$ are coordinate patches isomorphic to subsets in $\mathbb{R}^{n,k}$. For a treatment within the calculus of b-pseudodifferential operators, we refer the reader to [11]. To formulate the results proved in this paper, it suffices to work with the concept of totally characteristic pseudodifferential operators.

4. Integral operators

Let $\widetilde{\mathbb{X}}$ be the Oshima compactification of a Riemannian symmetric space $\mathbb{X} = G/K$ of non-compact type. As was already explained, G acts analytically on $\widetilde{\mathbb{X}}$, and the orbital decomposition is of normal crossing type. Consider the Banach space $\mathrm{C}(\widetilde{\mathbb{X}})$ of continuous, complex valued functions on $\widetilde{\mathbb{X}}$, equipped with the supremum norm, and let $(\pi, \mathrm{C}(\widetilde{\mathbb{X}}))$ be the corresponding continuous left-regular representation of G given by

$$\pi(g)\varphi(\tilde{x}) = \varphi\big(g^{-1} \cdot \tilde{x}\big), \quad \varphi \in \mathrm{C}(\widetilde{\mathbb{X}}).$$

The representation of the universal enveloping algebra $\mathfrak{U}$ of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ on the space of differentiable vectors $\mathrm{C}(\widetilde{\mathbb{X}})_\infty$ will be denoted by $d\pi$. We will also consider the regular representation of G on $\mathrm{C}^\infty(\widetilde{\mathbb{X}})$ which, equipped with the topology of uniform convergence, becomes a Fréchet space. This representation will be denoted by π as well. Let $(L, \mathrm{C}^\infty(G))$ be the left regular representation of G. With respect to the left-invariant metric on G given by $\langle\, ,\rangle_\theta$, we define $d(g,h)$ as the distance between two points $g, h \in G$, and set $|g| = d(g, e)$, where e is the identity element of G. A function f on G is said to be of *at most of exponential growth*, if there exists a $\kappa > 0$ such that $|f(g)| \leqslant Ce^{\kappa|g|}$ for some constant $C > 0$. As before, denote a Haar measure on G by d_G. Consider next the Casselman–Wallach space $\mathcal{S}(G)$ of rapidly decreasing functions on G introduced first in [21,7] in a slightly different way.

Definition 1. The space of rapidly decreasing functions on G, denoted by $\mathcal{S}(G)$, is given by all functions $f \in \mathrm{C}^\infty(G)$ satisfying the following conditions:

(i) For every $\kappa \geqslant 0$, and $X \in \mathfrak{U}$, there exists a constant $C > 0$ such that

$$\big|dL(X)f(g)\big| \leqslant Ce^{-\kappa|g|};$$

(ii) For every $\kappa \geqslant 0$, and $X \in \mathfrak{U}$, one has $dL(X)f \in \mathrm{L}^1(G, e^{\kappa|g|}d_G)$.

Remark 1.

(1) Note that condition (ii) in the previous definition is already implied by condition (i). Furthermore, if $f \in \mathcal{S}(G)$, $dR(X)f$ satisfies conditions (i) and (ii) of the definition as well.
(2) In our context, the consideration of the space $\mathcal{S}(G)$ was motivated by the study of strongly elliptic operators and the decay properties of the semigroups generated by them, see Section 5.

For later purposes, let us recall the following integration formulae.

Proposition 1. *Let $f_1 \in \mathcal{S}(G)$, and assume that $f_2 \in \mathrm{C}^\infty(G)$, together with all its derivatives, is at most of exponential growth. Let $X_1, \dots, X_d$ be a basis of $\mathfrak{g}$, and for $X^\gamma = X_{i_1}^{\gamma_1} \dots X_{i_r}^{\gamma_r}$ write $X^{\tilde{\gamma}} = X_{i_r}^{\gamma_r} \dots X_{i_1}^{\gamma_1}$, where γ is an arbitrary multi-index. Then*

$$\int_G f_1(g) dL\big(X^\gamma\big) f_2(g) d_G(g) = (-1)^{|\gamma|} \int_G dL\big(X^{\tilde{\gamma}}\big) f_1(g) f_2(g) d_G(g).$$

Proof. See [18, Proposition 1]. □

Next, we associate to every $f \in \mathcal{S}(G)$ and $\varphi \in \mathrm{C}(\widetilde{\mathbb{X}})$ the element $\int_G f(g)\pi(g)\varphi d_G(g) \in \mathrm{C}(\widetilde{\mathbb{X}})$. It is defined as a Bochner integral, and the continuous linear operator on $\mathrm{C}(\widetilde{\mathbb{X}})$ obtained this way is denoted by (1). Its restriction to $\mathrm{C}^\infty(\widetilde{\mathbb{X}})$ induces a continuous linear operator

$$\pi(f) : \mathrm{C}^\infty(\widetilde{\mathbb{X}}) \to \mathrm{C}^\infty(\widetilde{\mathbb{X}}) \subset \mathcal{D}'(\widetilde{\mathbb{X}}),$$

with Schwartz kernel given by the distribution section $\mathcal{K}_f \in \mathcal{D}'(\widetilde{\mathbb{X}} \times \widetilde{\mathbb{X}}, \mathbf{1} \boxtimes \Omega_{\widetilde{\mathbb{X}}})$. The properties of the Schwartz kernel $\mathcal{K}_f$ will depend on the analytic properties of f, as well as the orbit structure of the underlying G-action, and our main effort will be directed towards the elucidation of the structure of $\mathcal{K}_f$. For this, let us consider the orbital decomposition (4) of $\widetilde{\mathbb{X}}$, and remark that the restriction of $\pi(f)\varphi$ to any of the connected components isomorphic to $G/P_\Theta(K)$ depends only on the restriction of $\varphi \in \mathrm{C}(\widetilde{\mathbb{X}})$ to that component, so that we obtain the continuous linear operators

$$\pi(f)_{|\widetilde{\mathbb{X}}_\Theta} : \mathrm{C}^\infty_\mathrm{c}(\widetilde{\mathbb{X}}_\Theta) \to \mathrm{C}^\infty(\widetilde{\mathbb{X}}_\Theta),$$

where $\widetilde{\mathbb{X}}_\Theta$ denotes a component in $\widetilde{\mathbb{X}}$ isomorphic to $G/P_\Theta(K)$. Let us now assume that $\Theta = \Delta$, so that $P_\Theta(K) = K$. Since G acts transitively on $\widetilde{\mathbb{X}}_\Delta$ one deduces that $\pi(f)_{|\widetilde{\mathbb{X}}_\Delta} \in \mathrm{L}^{-\infty}(\widetilde{\mathbb{X}}_\Delta)$, c.p. [18, Section 4]. The main goal of this section is to prove that the restrictions of the operators $\pi(f)$ to the manifolds with corners $\overline{\widetilde{\mathbb{X}}_\Delta}$ are totally characteristic pseudodifferential operators of class $\mathrm{L}^{-\infty}_b$.

Let $\{(\widetilde{U}_{m_w}, \varphi_{m_w}^{-1})\}_{w \in W}$ be the finite atlas on the Oshima compactification $\widetilde{\mathbb{X}}$ defined earlier. For each point $\tilde{x} \in \widetilde{\mathbb{X}}$, choose open neighborhoods $\widetilde{W}_{\tilde{x}} \subset \widetilde{W}'_{\tilde{x}}$ of $\tilde{x}$ contained in a chart $\widetilde{U}_{m_w(\tilde{x})}$. Since $\widetilde{\mathbb{X}}$ is compact, we can find a finite subcover of the cover $\{\widetilde{W}_{\tilde{x}}\}_{\tilde{x} \in \widetilde{\mathbb{X}}}$, and in this way obtain a finite atlas $\{(\widetilde{W}_\gamma, \varphi_\gamma^{-1})\}_{\gamma \in I}$ of $\widetilde{\mathbb{X}}$, where for simplicity we wrote $\widetilde{W}_\gamma = \widetilde{W}_{\tilde{x}_\gamma}$, $\varphi_\gamma = \varphi_{m_w(\tilde{x}_\gamma)}$. Further, let $\{\alpha_\gamma\}_{\gamma \in I}$ be a partition of unity subordinate to this atlas, and let $\{\bar{\alpha}_\gamma\}_{\gamma \in I}$ be another set of functions satisfying $\bar{\alpha}_\gamma \in \mathrm{C}^\infty_\mathrm{c}(\widetilde{W}'_\gamma)$ and $\bar{\alpha}_{\gamma|\widetilde{W}_\gamma} \equiv 1$. Consider now the localization of $\pi(f)$ with respect to the atlas above given by

$$A_f^\gamma u = \big[\pi(f)_{|\widetilde{W}_\gamma}\big(u \circ \varphi_\gamma^{-1}\big)\big] \circ \varphi_\gamma, \quad u \in \mathrm{C}^\infty_\mathrm{c}(W_\gamma),\ W_\gamma = \varphi_\gamma^{-1}(\widetilde{W}_\gamma) \subset \mathbb{R}^{k+l}.$$

Writing $\varphi_\gamma^g = \varphi_\gamma^{-1} \circ g^{-1} \circ \varphi_\gamma$ and $x = (x_1, \dots, x_{k+l}) = (n,t) \in W_\gamma$ we obtain

$$A_f^\gamma u(x) = \int_G f(g)\big[\big(u \circ \varphi_\gamma^{-1}\big)\bar{\alpha}_\gamma\big]\big(g^{-1} \cdot \varphi_\gamma(x)\big) d_G(g) = \int_G f(g) c_\gamma(x,g)\big(u \circ \varphi_\gamma^g\big)(x) d_G(g),$$

where we put $c_\gamma(x,g) = \bar{\alpha}_\gamma(g^{-1} \cdot \varphi_\gamma(x))$. Next, define the functions

$$\hat{f}_\gamma(x,\xi) = \int_G e^{i\varphi_\gamma^g(x)\cdot\xi} c_\gamma(x,g) f(g) dg, \qquad a_f^\gamma(x,\xi) = e^{-ix\cdot\xi} \hat{f}_\gamma(x,\xi).$$

Differentiating under the integral we see that $\hat{f}_\gamma(x,\xi), a_f^\gamma(x,\xi) \in \mathrm{C}^\infty(W_\gamma \times \mathbb{R}^{k+l})$. We now have the following

Lemma 2. *For $\tilde{x} = \varphi_\gamma(n,t) \in \widetilde{W}_\gamma$, let $V_{\gamma,\tilde{x}}$ denote the set of $g \in G$ such that $g \cdot \tilde{x} \in \widetilde{W}_\gamma$. Then we have the power series expansion*

$$t_j(g \cdot \tilde{x}) = \sum_{\substack{\alpha,\beta \\ \beta_j \neq 0}} c^j_{\alpha,\beta}(g) n^\alpha(\tilde{x}) t^\beta(\tilde{x}), \quad j = 1, \dots, l, \tag{10}$$

where the coefficients $c^j_{\alpha,\beta}(g)$ depend real-analytically on $g \in V_{\gamma,\tilde{x}}$, and α, β are multi-indices.

Proof. By Theorem 1, a G-orbit in $\widetilde{\mathbb{X}}$ is locally determined by the signature of any of its elements. In particular, for $\tilde{x} \in \widetilde{W}_\gamma$ and $g \in V_{\gamma,\tilde{x}}$ as above, we have $\operatorname{sgn} t_j(g \cdot \tilde{x}) = \operatorname{sgn} t_j(\tilde{x})$ for all $j = 1, \dots, l$. Hence, $t_j(g \cdot \tilde{x}) = 0$ if and only if $t_j(\tilde{x}) = 0$. Now, due to the analyticity of the coordinates $(\varphi_\gamma, \widetilde{W}_\gamma)$, there is a power series expansion

$$t_j(g \cdot \tilde{x}) = \sum_{\alpha,\beta} c^j_{\alpha,\beta}(g) n^\alpha(\tilde{x}) t^\beta(\tilde{x}), \quad \tilde{x} \in \widetilde{W}_\gamma, \ g \in V_{\gamma,\tilde{x}},$$

for every $j = 1, \dots, l$, which can be rewritten as

$$t_j(g \cdot \tilde{x}) = \sum_{\substack{\alpha,\beta \\ \beta_j \neq 0}} c^j_{\alpha,\beta}(g) n^\alpha(\tilde{x}) t^\beta(\tilde{x}) + \sum_{\substack{\alpha,\beta \\ \beta_j = 0}} c^j_{\alpha,\beta}(g) n^\alpha(\tilde{x}) t^\beta(\tilde{x}). \tag{11}$$

Suppose $t_j(\tilde{x}) = 0$. Then the first summand of the last equation must vanish, as in each term of the summation a non-zero power of $t_j(\tilde{x})$ occurs. Also, $t_j(g \cdot \tilde{x}) = 0$. Therefore (11) implies that the second summand must vanish, too. But the latter is independent of t_j. So we conclude

$$\sum_{\substack{\alpha,\beta \\ \beta_j = 0}} c^j_{\alpha,\beta}(g) n^\alpha(\tilde{x}) t^\beta(\tilde{x}) \equiv 0$$

for all $\tilde{x} \in \widetilde{W}_\gamma$, $g \in V_{\gamma,\tilde{x}}$, and the assertion follows. $\square$

From Lemma 2 we deduce that

$$t_j(g\cdot\tilde{x}) = t_j^{q_j}(\tilde{x})\chi_j(g,\tilde{x}), \quad \tilde{x}\in\widetilde{W}_\gamma,\ g\in V_{\gamma,\tilde{x}},$$

where $\chi_j(g,\tilde{x})$ is a function that is real-analytic in g and in $\tilde{x}$, and $q_j \geqslant 1$ is the lowest power of t_j that occurs in the expansion (10). Furthermore, since $t_j(g\cdot\tilde{x}) = t_j(\tilde{x})$ for $g = e$, one has $q_1 = \cdots = q_l = 1$. A computation now shows that

$$1 = \chi_j\big(g^{-1}, g\cdot\tilde{x}\big)\cdot\chi_j(g,\tilde{x}), \quad \forall\tilde{x}\in\widetilde{W}_\gamma,\ g\in V_{\gamma,\tilde{x}},$$

where $g^{-1}\in V_{\gamma,g\tilde{x}}$. This implies

$$\chi_j(g,\tilde{x}) \neq 0, \quad \forall\tilde{x}\in\widetilde{W}_\gamma,\ g\in V_{\gamma,\tilde{x}}, \tag{12}$$

since $\chi_j(g^{-1}, g\cdot\tilde{x})$ is a finite complex number. Thus, for $\tilde{x} = \varphi_\gamma(x)\in\widetilde{W}_\gamma$, $x=(n,t)$, $g^{-1}\in V_{\gamma,\tilde{x}}$, we have

$$\varphi_\gamma^g(x) = \big(n_1\big(g^{-1}\cdot\tilde{x}\big),\dots,n_k\big(g^{-1}\cdot\tilde{x}\big), t_1(\tilde{x})\chi_1\big(g^{-1},\tilde{x}\big),\dots,t_l(\tilde{x})\chi_l\big(g^{-1},\tilde{x}\big)\big).$$

Note that similar formulae hold for $\tilde{x}\in\widetilde{U}_{m_w}$ and g sufficiently close to the identity. The following lemma describes the G-action on $\widetilde{\mathbb{X}}$ as far as the t-coordinates are concerned.

Lemma 3. *Let $X_{-\lambda,i}$ and H_j be the basis elements for $\mathfrak{n}^-$ and $\mathfrak{a}$ introduced in Section 2, $w\in W$, and $\tilde{x}\in\widetilde{U}_{m_w}$. Then, for small $s\in\mathbb{R}$,*

$$\chi_j\big(\mathrm{e}^{sH_i},\tilde{x}\big) = e^{-c_{ij}(m_w)s},$$

where the $c_{ij}(m_w)$ are the matrix coefficients of the adjoint representation of M^* on $\mathfrak{a}$, and are given by $\mathrm{Ad}(m_w^{-1})H_i = \sum_{j=1}^l c_{ij}(m_w)H_j$. Furthermore, when $\tilde{x} = \pi(e,n,t)$,

$$\chi_j\big(\mathrm{e}^{sX_{-\lambda,i}},\tilde{x}\big) \equiv 1.$$

Proof. Let $Y\in\mathfrak{g}$. From the proof of Lemma 1 it follows that the action of the one-parameter group $\exp(sY)$ on the homogeneous space $G/P_\Theta(K)$ is given by

$$\exp(sY)gnaP_\Theta(K) = gn\exp N_3^-(s)a\exp\big(A_1(s)+A_2(s)\big)P_\Theta(K), \tag{13}$$

where $N_3^-(s)\in\mathfrak{n}^-$, $A_1(s)\in\mathfrak{a}$, $A_2(s)\in\mathfrak{a}(\Theta)$. Denote the derivatives of $N_3^-(s)$, $A_1(s)$, and $A_2(s)$ at $s=0$ by N_3^-, A_1, and A_2 respectively. The analyticity of the G-action implies that $N_3^-(s)$, $A_1(s)$, $A_2(s)$ are real-analytic functions in s. Furthermore, from (13) it is clear that $N_3^-(0)=0$, $A_1(0)+A_2(0)=0$, so that for small s we have

$$A_1(s)+A_2(s)=(A_1+A_2)s+\frac{1}{2}\frac{d^2}{ds^2}\big(A_1(s)+A_2(s)\big)\bigg|_{s=0}s^2+\cdots,$$

$$N_3^-(s)=N_3^-s+\frac{1}{2}\frac{d^2}{ds^2}N_3^-(s)\bigg|_{s=0}s^2+\cdots.$$

Next, fix $m_w\in M^*$ and let $\Theta=\Delta$. The action of the one-parameter group corresponding to H_i at $\tilde{x}=\pi(m_w,n,t)\in\widetilde{U}_{m_w}\cap\widetilde{\mathbb{X}}_\Delta$ is given by

$$\exp(sH_i)m_wnaK=m_w\big(m_w^{-1}\exp(sH_i)m_w\big)naK=m_w\exp\big(s\,\mathrm{Ad}\big(m_w^{-1}\big)H_i\big)naK.$$

As m_w lies in M^*, $\exp(s\,\mathrm{Ad}(m_w^{-1})H_i)$ lies in A. Since A normalizes N^-, we conclude that $\exp(s\,\mathrm{Ad}(m_w^{-1})H_i)n\exp(-s\,\mathrm{Ad}(m_w^{-1})H_i)$ belongs to N^-. Writing

$$n^{-1}\exp\big(s\,\mathrm{Ad}\big(m_w^{-1}\big)H_i\big)n\exp\big(-s\,\mathrm{Ad}\big(m_w^{-1}\big)H_i\big)=\exp N_3^-(s)$$

we get

$$\exp(sH_i)m_wnaK=m_wn\exp N_3^-(s)a\exp\big(s\,\mathrm{Ad}\big(m_w^{-1}\big)H_i\big)K.$$

In the notation of (13) we therefore obtain $A_1(s)+A_2(s)=s\,\mathrm{Ad}(m_w^{-1})H_i$, and by writing $\mathrm{Ad}(m_w^{-1})H_i=\sum_{j=1}^{l}c_{ij}(m_w)H_j$ we arrive at

$$a\exp\big(A_1(s)+A_2(s)\big)=\exp\left(\sum_{j=1}^{l}\big(c_{ij}(m_w)s-\log t_j\big)H_j\right).$$

In terms of the coordinates this shows that $t_j(\exp(sH_i)\cdot\tilde{x})=t_j(\tilde{x})e^{-c_{ij}(m_w)s}$ for $\tilde{x}\in\widetilde{U}_{m_w}\cap\widetilde{\mathbb{X}}_\Delta$, and by analyticity we obtain that $\chi_j(\mathrm{e}^{sH_i},\tilde{x})=e^{-c_{ij}(m_w)s}$ for arbitrary $\tilde{x}\in\widetilde{U}_{m_w}$. On the other hand, let $Y=X_{-\lambda,i}$, and $\tilde{x}=\varphi_e(n,t)\in\widetilde{U}_e\cap\widetilde{\mathbb{X}}_\Delta$. Then the action corresponding to $X_{-\lambda,i}$ at $\tilde{x}$ is given by

$$\exp(sX_{-\lambda,i})naK=n\exp N_3^-(s)aK,$$

where we wrote $\exp N_3^-(s)=s\,\mathrm{Ad}(n^{-1})\exp X_{-\lambda,i}$. In terms of the coordinates this implies that $t_j(\exp(sX_{-\lambda,i})\cdot\tilde{x})=t_j(\tilde{x})$ showing that $\chi_j(\mathrm{e}^{sX_{-\lambda,i}},\tilde{x})\equiv 1$ for $\tilde{x}\in\widetilde{U}_e\cap\widetilde{\mathbb{X}}_\Delta$, and, by analyticity, for general $\tilde{x}\in\widetilde{U}_e$, finishing the proof of the lemma. □

Let now $x=(n,t)\in W_\gamma$, and let T_x be the diagonal $(l\times l)$-matrix with entries $x_{k+1},\dots,x_{k+l}$. Introduce the auxiliary symbol

$$\begin{aligned}\tilde{a}_f^\gamma(x,\xi)&=a_f^\gamma\big(x,\big(\mathbf{1}_k\otimes T_x^{-1}\big)\xi\big)=e^{-i(x_1,\dots,x_k,1,\dots,1)\cdot\xi}\int_G\psi_{\xi,x}^\gamma\big(g^{-1}\big)c_\gamma(x,g)f(g)d_G(g)\\&=\int_G e^{i\Psi_\gamma(g,x)\cdot\xi}c_\gamma(x,g)f(g)d_G(g),\end{aligned}\tag{14}$$

where we put

$$\begin{aligned}\Psi_\gamma(g,x) &= \big[\big(\mathbf{1}_k \otimes T_x^{-1}\big)\big(\varphi_\gamma^g(x) - x\big)\big] \\ &= \big(x_1\big(g^{-1}\cdot\tilde{x}\big) - x_1(\tilde{x}),\dots,x_k\big(g^{-1}\cdot\tilde{x}\big) - x_k(\tilde{x}),\chi_1\big(g^{-1},\tilde{x}\big) - 1, \\ &\quad \dots,\chi_l\big(g^{-1},\tilde{x}\big) - 1\big),\end{aligned}$$

as well as

$$\psi_{\xi,x}^\gamma(g) = e^{i(x_1(g\cdot\tilde{x}),\dots,x_k(g\cdot\tilde{x}),\chi_1(g,\tilde{x}),\dots,\chi_l(g,\tilde{x}))\cdot\xi}.$$

Clearly, $\tilde{a}_f^\gamma(x,\xi) \in \mathrm{C}^\infty(W_\gamma \times \mathbb{R}^{k+l})$. Our next goal is to show that $\tilde{a}_f^\gamma(x,\xi)$ is a lacunary symbol. To do so, we need the following

Proposition 2. *Let $(L, \mathrm{C}^\infty(G))$ be the left regular representation of G. Let $X_{-\lambda,i}, H_j$ be the basis elements of $\mathfrak{n}^-$ and $\mathfrak{a}$ introduced in Section 2, and $(\widetilde{W}_\gamma,\varphi_\gamma)$ an arbitrary chart. With $x=(n,t)\in W_\gamma$, $\tilde{x}=\varphi_\gamma(x)\in\widetilde{W}_\gamma$, $g\in V_{\gamma,\tilde{x}}$ one has*

$$\begin{pmatrix} dL(X_{-\lambda,1})\psi_{\xi,x}^\gamma(g) \\ \vdots \\ dL(H_l)\psi_{\xi,x}^\gamma(g)\end{pmatrix} = i\psi_{\xi,x}^\gamma(g)\Gamma(x,g)\xi, \tag{15}$$

with

$$\Gamma(x,g) = \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_3 & \Gamma_4\end{pmatrix} = \left(\begin{array}{c|c} dL(X_{-\lambda,i})n_{j,\tilde{x}}(g) & dL(X_{-\lambda,i})\chi_j(g,\tilde{x}) \\ \hline dL(H_i)n_{j,\tilde{x}}(g) & dL(H_i)\chi_j(g,\tilde{x})\end{array}\right) \tag{16}$$

belonging to $\mathrm{GL}(l+k,\mathbb{R})$, where $n_{j,\tilde{x}}(g) = n_j(g\cdot\tilde{x})$.

Proof. Fix a chart $(\widetilde{W}_\gamma,\varphi_\gamma^{-1})$, and let x, $\tilde{x}$, g be as above. For $X\in\mathfrak{g}$, one computes that

$$\begin{aligned} dL(X)\psi_{\xi,x}^\gamma(g) &= \frac{d}{ds} e^{i(\mathbf{1}_k\otimes T_x^{-1})\varphi_\gamma^{\mathrm{e}^{-sX}g}(x)\cdot\xi}\Big|_{s=0} \\ &= i\psi_{\xi,x}^\gamma(g)\Bigg[\sum_{i=1}^k \xi_i dL(X)n_{i,\tilde{x}}(g) + \sum_{j=1}^l \xi_{k+j} dL(X)\chi_j(g,\tilde{x})\Bigg],\end{aligned}$$

showing the first equality. To see the invertibility of the matrix $\Gamma(x,g)$, note that for small s

$$\chi_j\big(\mathrm{e}^{-sX}g,\tilde{x}\big) = \chi_j(g,\tilde{x})\chi_j\big(\mathrm{e}^{-sX},g\cdot\tilde{x}\big).$$

Lemma 3 then yields

$$dL(H_i)\chi_j(g,\tilde{x}) = \chi_j(g,\tilde{x})\frac{d}{ds}\big(e^{c_{ij}(m_{w_\gamma})s}\big)_{|s=0} = \chi_j(g,\tilde{x})c_{ij}(m_{w_\gamma}).$$

This means that Γ_4 is the product of the matrix $(c_{ij}(m_{w_\gamma}))_{i,j}$ with the diagonal matrix whose j-th diagonal entry is $\chi_j(g,\tilde{x})$. Since $(c_{ij}(m_{w_\gamma}))_{i,j}$ is just the matrix realization of $\mathrm{Ad}(m_{w_\gamma}^{-1})$ relative to the basis $\{H_1,\ldots,H_l\}$ of $\mathfrak{a}$, it is invertible. On the other hand, by (12), $\chi_j(g,\tilde{x})$ is non-zero for all $j \in \{1,\ldots,l\}$ and arbitrary g and $\tilde{x}$. Therefore Γ_4, being the product of two invertible matrices, is invertible. Next, let us show that the matrix Γ_1 is non-singular. Its (ij)th entry reads

$$dL(X_{-\lambda,i})n_{j,\tilde{x}}(g) = \frac{d}{ds} n_{j,\tilde{x}}\big(\mathrm{e}^{-sX_{-\lambda,i}} \cdot g\big)_{|s=0} = (-X_{-\lambda,i|\widetilde{\mathbb{X}}})_{g\cdot\tilde{x}}(n_j).$$

For $\Theta \subset \Delta$, $q \in \mathbb{R}^l$, we define the k-dimensional submanifolds

$$\mathfrak{L}_\Theta(q) = \big\{\tilde{x} = \varphi_\gamma(n,q) \in \widetilde{W}_\gamma : q_i \neq 0 \Leftrightarrow \alpha_i \in \Theta\big\}.$$

As g varies over G in Lemma 1, one deduces that $N^- \times A(\Theta)$ acts locally transitively on $\widetilde{\mathbb{X}}_\Theta$. In addition, $T_{g\cdot\tilde{x}}\mathfrak{L}_\Theta(q)$ is equal to the span of the vector fields $\{X_{-\lambda,i|\widetilde{\mathbb{X}}}\}$, which means that N^- acts locally transitively on $\mathfrak{L}_\Theta(q)$ for arbitrary Θ. Since the latter is parametrized by the coordinates $(n_1,\ldots,n_k)$, one concludes that the matrix $((X_{-\lambda,i|\widetilde{\mathbb{X}}})_{g\cdot\tilde{x}}(n_j))_{ij}$ has full rank. Thus, Γ_1 is non-singular. On the other hand, if $\tilde{x} = \pi(e,n,t) \in \widetilde{U}_e$, Lemma 3 implies

$$dL(X_{-\lambda,i})\chi_j(g,\tilde{x}) = \chi_j(g,\tilde{x})\frac{d}{ds}\big(\chi_j\big(e^{-sX_{-\lambda,i}}, g\cdot\tilde{x}\big)\big)_{|s=0} = 0,$$

showing that Γ_2 is identically zero, while Γ_4 is a non-singular diagonal matrix in this case. Geometrically, this amounts to the fact that the fundamental vector field corresponding to H_j is transversal to the hypersurface defined by $t_j = q \in \mathbb{R}\backslash\{0\}$, while the vector fields corresponding to the Lie algebra elements $X_{-\lambda,r}, H_i$, $i \neq j$, are tangential. We therefore conclude that $\Gamma(x,g)$ is non-singular if $\tilde{x} \in \widetilde{U}_e$, which is dense in $\widetilde{\mathbb{X}}$. For symmetry reasons, the same must hold if $\tilde{x}$ lies in one of the remaining charts $\widetilde{U}_{m_{w_\gamma}}$, and the assertion of the proposition follows. □

We can now state the main result of this paper. In what follows, $\{(\widetilde{W}_\gamma, \varphi_\gamma^{-1})\}_{\gamma\in I}$ will always denote the finite atlas of $\widetilde{\mathbb{X}}$ constructed above.

Theorem 2. *Let $\widetilde{\mathbb{X}}$ be the Oshima compactification of a Riemannian symmetric space $\mathbb{X} = G/K$ of non-compact type, and $f \in \mathcal{S}(G)$ a rapidly decaying function on G. Then the operators $\pi(f)$ are locally of the form*

$$A_f^\gamma u(x) = \int e^{ix\cdot\xi} a_f^\gamma(x,\xi)\hat{u}(\xi)d\xi, \quad u \in \mathrm{C_c^\infty}(W_\gamma), \tag{17}$$

where $a_f^\gamma(x,\xi) = \tilde{a}_f^\gamma(x,\xi_1,\ldots,\xi_k,x_{k+1}\xi_{k+1},\ldots,\xi_{k+l}x_{k+l})$, and $\tilde{a}_f^\gamma(x,\xi) \in \mathrm{S}_{la}^{-\infty}(W_\gamma \times \mathbb{R}_\xi^{k+l})$ is given by (14). In particular, the kernel of the operator A_f^γ is determined by its *restrictions to* $W_\gamma^* \times W_\gamma^*$, *where* $W_\gamma^* = \{x = (n,t) \in W_\gamma : t_1 \cdots t_l \neq 0\}$, *and given by the oscillatory integral*

$$K_{A_f^\gamma}(x,y) = \int e^{i(x-y)\cdot\xi} a_f^\gamma(x,\xi) đ\xi. \tag{18}$$

As a consequence, we obtain the following

Corollary 1. *Let* $\widetilde{\mathbb{X}}_\Delta$ *be an open* G*-orbit in* $\widetilde{\mathbb{X}}$ *isomorphic to* G/K. *Then the continuous linear operators*

$$\pi(f)_{|\overline{\widetilde{\mathbb{X}}_\Delta}} : \mathrm{C}_c^\infty(\overline{\widetilde{\mathbb{X}}_\Delta}) \to \mathrm{C}^\infty(\overline{\widetilde{\mathbb{X}}_\Delta}),$$

are totally characteristic pseudodifferential operators of class $\mathrm{L}_b^{-\infty}$ *on the manifold with corner* $\overline{\widetilde{\mathbb{X}}_\Delta}$.

Proof of Theorem 2. Our considerations closely follow, by adapting to our setting, the reasoning of the proof of Theorem 4 in [18]. Let $\Gamma(x,g)$ be the matrix defined in (16), and consider its extension as an endomorphism in $\mathbb{C}^1[\mathbb{R}_\xi^{k+l}]$ to the symmetric algebra $\mathrm{S}(\mathbb{C}^1[\mathbb{R}_\xi^{k+l}]) \simeq \mathbb{C}[\mathbb{R}_\xi^{k+l}]$. By Proposition 2, $\Gamma(x,g)$ is invertible for $\tilde{x} \in \widetilde{W}_\gamma$, $g \in V_{\gamma,\tilde{x}}$. Therefore, its extension to $\mathrm{S}^N(\mathbb{C}^1[\mathbb{R}_\xi^{k+l}])$ is also an automorphism for any $N \in \mathbb{N}$. Regarding the polynomials $\xi_1, \dots, \xi_{k+l}$ as a basis in $\mathbb{C}^1[\mathbb{R}_\xi^{k+l}]$, let us denote the image of the basis vector ξ_j under the endomorphism $\Gamma(x,g)$ by $\Gamma\xi_j$, so that by (15)

$$\begin{aligned} \Gamma\xi_j &= -i\psi_{-\xi,x}^\gamma(g) dL(X_{-\lambda,j}) \psi_{\xi,x}^\gamma(g), \quad 1 \leqslant j \leqslant k, \\ \Gamma\xi_j &= -i\psi_{-\xi,x}^\gamma(g) dL(H_j) \psi_{\xi,x}^\gamma(g), \quad k+1 \leqslant j \leqslant k+l. \end{aligned}$$

Every polynomial $\xi_{j_1} \otimes \cdots \otimes \xi_{j_N} \equiv \xi_{j_1} \dots \xi_{j_N}$ can then be written as a linear combination

$$\xi^\alpha = \sum_\beta \Lambda_\beta^\alpha(x,g) \Gamma\xi_{\beta_1} \cdots \Gamma\xi_{\beta_{|\alpha|}}, \tag{19}$$

where the $\Lambda_\beta^\alpha(x,g)$ are real-analytic functions given in terms of the matrix coefficients of $\Gamma(x,g)$. We need now the following

Lemma 4. *For arbitrary indices* $\beta_1, \dots, \beta_r$, *one has*

$$\begin{aligned} i^r \psi_{\xi,x}^\gamma(g) \Gamma\xi_{\beta_1} \cdots \Gamma\xi_{\beta_r} &= dL(X_{\beta_1} \cdots X_{\beta_r}) \psi_{\xi,x}^\gamma(g) \\ &\quad + \sum_{s=1}^{r-1} \sum_{\alpha_1,\dots,\alpha_s} d_{\alpha_1,\dots,\alpha_s}^{\beta_1,\dots,\beta_r}(x,g) dL(X_{\alpha_1} \cdots X_{\alpha_s}) \psi_{\xi,x}^\gamma(g), \end{aligned} \tag{20}$$

where the coefficients $d_{\alpha_1,\dots,\alpha_s}^{\beta_1,\dots,\beta_r}(x,g)$ *are real-analytic functions given by the matrix coefficients of* $\Gamma(x,g)$ *which are at most of exponential growth in* g, *and independent of* ξ.

Proof. The lemma is proved by induction. For $r = 1$ one has $i\psi^\gamma_{\xi,x}(g)\Gamma\xi_p = dL(X_p)\psi^\gamma_{\xi,x}(g)$, where $1 \leqslant p \leqslant d$. Differentiating the latter equation with respect to X_j, and writing $\Gamma\xi_p = \sum_{s=1}^{k+l}\Gamma_{ps}(x,g)\xi_s$, we obtain with (19) the equality

$$-\psi^\gamma_{\xi,x}(g)\Gamma\xi_j\Gamma\xi_p = dL(X_jX_p)\psi^\gamma_{\xi,x}(g) - \sum_{s,r=1}^{k+l}\bigl(dL(X_j)\Gamma_{ps}\bigr)(x,g)\Lambda^s_r(x,g)dL(X_r)\psi^\gamma_{\xi,x}(g).$$

Hence, the assertion of the lemma is correct for $r = 1, 2$. Now, assume that it holds for $r \leqslant N$. Setting $r = N$ in (20), and differentiating with respect to X_p, yields for the left hand side

$$\begin{aligned}&i^{N+1}\psi^\gamma_{\xi,x}(g)\Gamma\xi_p\Gamma\xi_{\beta_1}\cdots\Gamma\xi_{\beta_N}\\&\quad + i^N\psi^\gamma_{\xi,x}(g)\left(\sum_{s,q=1}^{k+l}\bigl(dL(X_p)\Gamma_{\beta_1 s}\bigr)(x,g)\Lambda^s_q(x,g)\Gamma\xi_q\right)\Gamma\xi_{\beta_2}\cdots\Gamma\xi_{\beta_N} + \cdots.\end{aligned}$$

By assumption, we can apply (20) to the products $\Gamma\xi_q\Gamma\xi_{\beta_2}\cdots\Gamma\xi_{\beta_N},\ldots$ of at most N factors. Since

$$\bigl\|\pi(g)\bigr\| \leqslant ce^{\kappa|g|}, \quad g \in G, \tag{21}$$

for some constants $c \geqslant 1$, $\kappa \geqslant 0$, see [19, p. 12], the functions $n_{i,\tilde{x}}(g)$, $\chi_j(g,\tilde{x})$, and consequently the coefficients of $\Gamma(x,g)$, are at most of exponential growth in g, and the assertion of the lemma follows. $\square$

End of proof of Theorem 2. Let us next show that $\tilde{a}^\gamma_f(x,\xi) \in \mathrm{S}^{-\infty}(W_\gamma \times \mathbb{R}^{k+l}_\xi)$. As already noted, $\tilde{a}^\gamma_f(x,\xi) \in \mathrm{C}^\infty(W_\gamma \times \mathbb{R}^{k+l}_\xi)$. While differentiation with respect to ξ does not alter the growth properties of $\tilde{a}^\gamma_f(x,\xi)$, differentiation with respect to x yields additional powers in ξ. Now, as an immediate consequence of (19) and (20), one computes for arbitrary $N \in \mathbb{N}$

$$\psi^\gamma_{\xi,x}(g)\bigl(1+|\xi|^2\bigr)^N = \sum_{r=0}^{2N}\sum_{|\alpha|=r} b^N_\alpha(x,g)dL\bigl(X^\alpha\bigr)\psi^\gamma_{\xi,x}(g), \tag{22}$$

where the coefficients $b^N_\alpha(x,g)$ are at most of exponential growth in g. Now, $(\partial^\alpha_\xi\partial^\beta_x\tilde{a}^\gamma_f) \times (x,\xi)$ is a finite sum of terms of the form

$$\xi^{\beta'}e^{-i(x_1,\ldots,x_k,1,\ldots,1)\cdot\xi}\int_G f(g)d^\alpha_{\beta'\beta''}(x,g)\psi^\gamma_{\xi,x}\bigl(g^{-1}\bigr)\bigl(\partial^{\beta''}_x c_\gamma\bigr)(x,g)dg,$$

the functions $d^\alpha_{\beta'\beta''}(x,g)$ being at most of exponential growth in g. Making use of (22), and integrating according to Proposition 1, we finally obtain for arbitrary α, β the estimate

$$\left|\left(\partial_\xi^\alpha \partial_x^\beta \tilde{a}_f^\gamma\right)(x,\xi)\right| \leqslant \frac{1}{(1+\xi^2)^N} C_{\alpha,\beta,\mathcal{K}} \quad x \in \mathcal{K},$$

where $\mathcal{K}$ denotes an arbitrary compact set in W_γ, and $N \in \mathbb{N}$. This proves that $\tilde{a}_f^\gamma(x,\xi) \in \mathrm{S}^{-\infty}(W_\gamma \times \mathbb{R}_\xi^{k+l})$. Since (17) is an immediate consequence of the Fourier inversion formula, it remains to show that $\tilde{a}_f^\gamma(x,\xi)$ satisfies the lacunary condition (8) for each of the coordinates t_i. Now, it is clear that $a_f^\gamma(x,\xi) \in \mathrm{S}^{-\infty}(W_\gamma^* \times \mathbb{R}_\xi^{k+l})$, since G acts transitively on each $\widetilde{\mathbb{X}}_\Delta$. As a consequence, the Schwartz kernel of the restriction of the operator $A_f^\gamma : \mathrm{C}_\mathrm{c}^\infty(W_\gamma) \to \mathrm{C}^\infty(W_\gamma)$ to W_γ^* is given by the absolutely convergent integral

$$\int e^{i(x-y)\cdot\xi} a_f^\gamma(x,\xi)\,d\xi \in \mathrm{C}^\infty\left(W_\gamma^* \times W_\gamma^*\right).$$

Next, let us write $W_\gamma = \bigcup_{\Theta \subset \Delta} W_\gamma^\Theta$, where $W_\gamma^\Theta = \{x = (n,t) : t_i \neq 0 \Leftrightarrow \alpha_i \in \Theta\}$. Since on W_γ^Θ the function $A_f^\gamma u$ depends only on the restriction of $u \in \mathrm{C}_\mathrm{c}^\infty(W_\gamma)$ to W_γ^Θ, one deduces that

$$\operatorname{supp} K_{A_f^\gamma} \subset \bigcup_{\Theta \subset \Delta} \overline{W_\gamma^\Theta} \times \overline{W_\gamma^\Theta}. \tag{23}$$

Therefore, each of the integrals

$$\int e^{i(x_j-y_j)\xi_j} \tilde{a}_f^\gamma\big(x, (\mathbf{1}_k \otimes T_x)\xi\big)\,d\xi_j, \quad j = k+1,\dots,k+l,$$

which are smooth functions on $W_\gamma^* \times W_\gamma^*$, must vanish if x_j and y_j do not have the same sign. With the substitution $r_j = y_j/x_j - 1$, $\xi_j x_j = \xi_j'$ one finally arrives at the conditions

$$\int e^{-ir_j\xi_j} \tilde{a}_f^\gamma(x,\xi)\,d\xi_j = 0 \quad \text{for } r_j < -1,\ x \in W_\gamma^*.$$

But since $\tilde{a}_f^\gamma$ is rapidly decreasing in ξ, the Lebesgue bounded convergence theorem implies that these conditions must also hold for $x \in W_\gamma$. Thus, the lacunarity of the symbol $\tilde{a}_f^\gamma$ follows. The fact that the kernel $K_{A_f^\gamma}$ must be determined by its restriction to $W_\gamma^* \times W_\gamma^*$, and hence by the oscillatory integral (18) now follows by arguments analogous to those given in [15, Lemma 4.1]. This completes the proof of Theorem 2. □

As a consequence of Theorem 2, we can describe the asymptotic behavior of the kernels $K_{A_f^\gamma}(x,y)$ as $|x_j| \to 0$ or $|y_j| \to 0$ for $k+1 \leqslant j \leqslant k+l$. Note that this corresponds to the asymptotic behavior of the kernel of $\pi(f)$ on $\widetilde{\mathbb{X}}_\Delta \simeq \mathbb{X}$ at infinity.

Corollary 2. *Let $k+1 \leqslant j \leqslant k+l$. Then $K_{A_f^\gamma}(x,y)$ is rapidly decreasing as $|x_j| \to 0$ or $|y_j| \to 0$, provided that $x_j \neq y_j$.*

Proof. According to Theorem 2, the kernel of $\pi(f)$ is locally given by

$$K_{A_f^\gamma}(x,y) = \int e^{i(x-y)\cdot\xi} a_f^\gamma(x,\xi) đ\xi = \int e^{i(x-y)\cdot(\mathbf{1}_k \otimes T_x^{-1})\xi} \tilde{a}_f^\gamma(x,\xi) \left|\det\left(\mathbf{1}_k \otimes T_x^{-1}\right)'(\xi)\right| đ\xi$$
$$= \frac{1}{|x_{k+1}\cdots x_{k+l}|} \tilde{A}_f^\gamma\left(x, x_1 - y_1, \ldots, 1 - \frac{y_{k+1}}{x_{k+1}}, \ldots\right), \quad x_{k+1}\cdots x_{k+l} \neq 0,$$

where $\tilde{A}_f^\gamma(x,y)$ denotes the inverse Fourier transform of $\tilde{a}_f^\gamma(x,\xi)$,

$$\tilde{A}_f^\gamma(x,y) = \int e^{iy\cdot\xi} \tilde{a}_f^\gamma(x,\xi) đ\xi. \tag{24}$$

Since for $x \in W^\gamma$ the amplitude $\tilde{a}_f^\gamma(x,\xi)$ is rapidly falling in ξ, it follows that $\tilde{A}_f^\gamma(x,y) \in \mathcal{S}(\mathbb{R}_y^{k+l})$, the Fourier transform being an isomorphism on Schwartz space. Therefore the kernel $K_{A_f^\gamma}(x,y)$ is rapidly decreasing as $|x_j| \to 0$ if $x_j \neq y_j$ and $k+1 \leqslant j \leqslant k+l$. Furthermore, by the lacunarity of $\tilde{a}_f^\gamma$, $K_{A_f^\gamma}(x,y)$ is also rapidly decaying as $|y_j| \to 0$ if $x_j \neq y_j$ and $k+1 \leqslant j \leqslant k+l$. $\square$

The explicit local form of the kernels of $\pi(f)$ in the above proof shows that the singularities arise precisely from the lower-dimensional orbits, which are given by the vanishing of one or more of the coordinates $x_{k+1}, \ldots, x_{k+l}$.

5. Holomorphic semigroup and resolvent kernels

In this section, we study the holomorphic semigroup generated by a strongly elliptic operator Ω associated to the regular representation $(\pi, \mathrm{C}(\widetilde{\mathbb{X}}))$ of G, as well as its resolvent. Both the holomorphic semigroup and the resolvent can be characterized as convolution operators of the type considered before, so that we can study them by the methods developed in the previous section. In particular, this will allow us to obtain a description of the asymptotic behavior of the semigroup and resolvent kernels on $\widetilde{\mathbb{X}}_\Delta \simeq \mathbb{X}$ at infinity.

Let us begin by recalling some basic facts about elliptic operators and parabolic evolution equations on Lie groups, our main reference being [19]. Let $\mathcal{G}$ be a Lie group, and π a continuous representation of $\mathcal{G}$ on a Banach space $\mathcal{B}$. Let further $X_1, \ldots, X_d$ be a basis of the Lie algebra $\mathrm{Lie}(\mathcal{G})$ of $\mathcal{G}$, and

$$\Omega = \sum_{|\alpha| \leqslant q} c_\alpha d\pi\left(X^\alpha\right)$$

a *strongly elliptic differential operator of order* q associated with π, meaning that for all $\xi \in \mathbb{R}^d$ one has the inequality $\mathrm{Re}(-1)^{q/2} \sum_{|\alpha|=q} c_\alpha \xi^\alpha \geqslant \kappa|\xi|^q$ for some $\kappa > 0$. By the general theory of strongly continuous semigroups, its closure generates a strongly continuous holomorphic semigroup of bounded operators given by

$$S_\tau = \frac{1}{2\pi i}\int_\Gamma e^{\lambda\tau}(\lambda\mathbf{1}+\overline{\Omega})^{-1}d\lambda,$$

where Γ is an appropriate path in $\mathbb{C}$ coming from infinity and going to infinity such that λ does not lie in the spectrum $\sigma(\overline{\Omega})$ of $\overline{\Omega}$ for $\lambda \in \Gamma$. Here $|\arg\tau| < \alpha$ for an appropriate $\alpha \in (0, \pi/2]$, and the integral converges uniformly with respect to the operator norm. Furthermore, for $\tau > 0$, the subgroup S_τ can be characterized by a convolution semigroup of complex measures $\{\mu_\tau\}_{\tau>0}$ on $\mathcal{G}$ according to

$$S_\tau = \int_{\mathcal{G}} \pi(g)d\mu_\tau(g),$$

π being measurable with respect to the measures μ_τ. The measures μ_τ are absolutely continuous with respect to Haar measure $d_{\mathcal{G}}$ on $\mathcal{G}$, and denoting by $f_\tau(g) \in L^1(\mathcal{G}, d_{\mathcal{G}})$ the corresponding Radon–Nikodym derivative, one has

$$S_\tau = \pi(f_\tau) = \int_{\mathcal{G}} f_\tau(g)\pi(g)d_{\mathcal{G}}(g).$$

The function $f_\tau(g) \in \mathrm{L}^1(\mathcal{G}, d_{\mathcal{G}})$ is analytic in τ and g, and universal for all Banach representations. It satisfies the parabolic differential equation

$$\frac{\partial f_\tau}{\partial\tau}(g) + \sum_{|\alpha|\leqslant q} c_\alpha dL\big(X^\alpha\big)f_\tau(g) = 0, \qquad \lim_{\tau\to 0} f_\tau(g) = \delta(g),$$

where $(L, \mathrm{C}^\infty(\mathcal{G}))$ denotes the left regular representation of $\mathcal{G}$. As a consequence, f_τ must be supported on the identity component $\mathcal{G}_0$ of $\mathcal{G}$. We call it the *Langlands kernel* of the holomorphic semigroup S_τ, and it satisfies the following L^1- and L^∞-bounds.

Theorem 3. *For each $\kappa \geqslant 0$, there exist constants $a, b, c > 0$, and $\omega \geqslant 0$ such that*

$$\int_{\mathcal{G}_0} \big|dL\big(X^\alpha\big)\partial_\tau^\beta f_\tau(g)\big|e^{\kappa|g|}d_{\mathcal{G}_0}(g) \leqslant ab^{|\alpha|}c^\beta|\alpha|!\beta!\big(1+\tau^{-\beta-|\alpha|/q}\big)e^{\omega\tau}, \tag{25}$$

for all $\tau > 0$, $\beta = 0, 1, 2, \dots$ and multi-indices α. Furthermore,

$$\big|dL\big(X^\alpha\big)\partial_\tau^\beta f_\tau(g)\big| \leqslant ab^{|\alpha|}c^\beta|\alpha|!\beta!\big(1+\tau^{-\beta-(|\alpha|+d+1)/q}\big)e^{\omega\tau}e^{-\kappa|g|}, \tag{26}$$

for all $g \in \mathcal{G}_0$, where $d = \dim\mathcal{G}_0$, and q denotes the order of Ω. Similar bounds hold for the derivatives $dR(X^\alpha)$.

A detailed exposition of these facts can be found in [19, pp. 30, 152, 166, and 167]. Let now $\mathcal{G} = G$, and $(\pi, \mathcal{B})$ be the regular representation of G on $C(\widetilde{\mathbb{X}})$. Theorem 3 implies that the Langlands kernel f_τ belongs to the space $\mathcal{S}(G)$ of rapidly falling functions on G. As a consequence of the previous considerations we obtain

Theorem 4. *Let Ω be a strongly elliptic differential operator of order q associated with the regular representation $(\pi, C(\widetilde{\mathbb{X}}))$, and $S_\tau = \pi(f_\tau)$ the holomorphic semigroup of bounded operators generated by $\overline{\Omega}$. Then the operators S_τ are locally of the form (17) with f being replaced by f_τ, and totally characteristic pseudodifferential operators of class $\mathrm{L}_b^{-\infty}$ on the manifolds with corners $\overline{\widetilde{\widetilde{\mathbb{X}}}_\Delta}$. Furthermore, on $W_\gamma \times W_\gamma$, the kernel of S_τ is given by*

$$S_\tau^\gamma(x,y) = K_{A^\gamma_{f_\tau}}(x,y) = \int e^{i(x-y)\cdot\xi} a^\gamma_{f_\tau}(x,\xi) đ\xi$$
$$= \frac{1}{|x_{k+1}\cdots x_{k+l}|} \tilde{A}^\gamma_{f_\tau}\big(x, \big(\mathbf{1}_k \otimes T_x^{-1}\big)(x-y)\big),$$

where $x_{k+1}\cdots x_{k+l} \neq 0$, and $\tilde{A}^\gamma_{f_\tau}(x,y)$ was defined in (24). In particular, $S_\tau^\gamma(x,y)$ is rapidly falling as $|x_j| \to 0$, or $|y_j| \to 0$, as long as $x_j \neq y_j$, where $k+1 \leqslant j \leqslant k+l$. In addition,

$$\big|\tilde{A}^\gamma_{f_\tau}(x,y)\big| \leqslant \begin{cases} c_1(1+\tau^{-(l+k+1)/q}), & 0 < \tau \leqslant 1, \\ c_2 e^{\omega\tau}, & 1 < \tau, \end{cases} \tag{27}$$

uniformly on compact subsets of $W_\gamma \times W_\gamma$ for some constants $c_i > 0$.

Proof. The first assertions are immediate consequences of Theorem 2, and its corollary. In order to prove (27), note that for large $N \in \mathbb{N}$ one computes with (14), (22), and (24)

$$\begin{aligned}
&\big|\tilde{A}^\gamma_{f_\tau}(x,y)\big| \\
&\quad\leqslant \int \big|\tilde{a}^\gamma_{f_\tau}(x,\xi)\big| đ\xi = \int \bigg| \int_G \psi^\gamma_{\xi,x}\big(g^{-1}\big) c_\gamma(x,g) f_\tau(g) d_G(g) \bigg| đ\xi \\
&\quad= \int \big(1+|\xi|^2\big)^{-N} \bigg| \int_G c_\gamma(x,g) f_\tau(g) \sum_{r=0}^{2N} \sum_{|\alpha|=r} b^N_\alpha\big(x,g^{-1}\big) dL\big(X^\alpha\big) \psi^\gamma_{\xi,x}\big(g^{-1}\big) d_G(g) \bigg| đ\xi.
\end{aligned}$$

If we now apply Proposition 1, and take into account the estimate (25) we obtain

$$\begin{aligned}
&\big|\tilde{A}^\gamma_{f_\tau}(x,y)\big| \\
&\quad\leqslant \int \big(1+|\xi|^2\big)^{-N} \bigg| \int_G \psi^\gamma_{\xi,x}(g) \sum_{r=0}^{2N} \sum_{|\alpha|=r} dL\big(X^{\tilde{\alpha}}\big) \big[b^N_\alpha(x,g) c_\gamma\big(x,g^{-1}\big) f_\tau\big(g^{-1}\big)\big] d_G(g) \bigg| đ\xi \\
&\quad\leqslant \begin{cases} c_1(1+\tau^{-2N/q}), & 0 < \tau \leqslant 1, \\ c_2 e^{\omega\tau}, & 1 < \tau, \end{cases}
\end{aligned}$$

for certain constants $c_i > 0$. Expressing $\xi_j^{k+l+1} \psi^\gamma_{\xi,x}(g)$ on $\{\xi \in \mathbb{R}^n : |\xi_i| \leqslant |\xi_j| \text{ for all } i\}$ as left derivatives of $\psi^\gamma_{\xi,x}(g)$ according to (19) and (20), and estimating the maximum norm by the usual norm, a similar argument shows that the last estimate is also valid for $N = (k+l+1)/2$, compare (33). The proof is now complete. $\square$

Let us now turn to the resolvent of the closure of the strongly elliptic operator Ω. By (25) one has the bound $\|S_\tau\| \leqslant ce^{\omega\tau}$ for some constants $c \geqslant 1$, $\omega \geqslant 0$. For $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$, the resolvent of $\overline{\Omega}$ can be expressed by means of the Laplace transform according to

$$(\lambda \mathbf{1} + \overline{\Omega})^{-1} = \Gamma(1)^{-1} \int_0^\infty e^{-\lambda\tau} S_\tau d\tau,$$

where Γ is the Γ-function. More generally, one can consider for arbitrary $\alpha > 0$ the integral transforms

$$(\lambda \mathbf{1} + \overline{\Omega})^{-\alpha} = \Gamma(\alpha)^{-1} \int_0^\infty e^{-\lambda\tau} \tau^{\alpha-1} S_\tau d\tau.$$

As it turns out, the functions

$$r_{\alpha,\lambda}(g) = \Gamma(\alpha)^{-1} \int_0^\infty e^{-\lambda\tau} \tau^{\alpha-1} f_\tau(g) d\tau$$

are in $\mathrm{L}^1(G, e^{\kappa|g|} d_G)$, where $\kappa \geqslant 0$ is such that $\|\pi(g)\| \leqslant ce^{\kappa|g|}$ for some $c \geqslant 1$, see (21). This implies that the resolvent of $\overline{\Omega}$ can be expressed as the convolution operator

$$(\lambda \mathbf{1} + \overline{\Omega})^{-\alpha} = \pi(r_{\alpha,\lambda}) = \int_G r_{\alpha,\lambda}(g) \pi(g) d_G(g).$$

The resolvent kernels $r_{\alpha,\lambda}$ decrease exponentially as $|g| \to \infty$, but they are singular at the identity if $d \geqslant q\alpha$. More precisely, one has the following

Theorem 5. *There exist constants $b, c, \lambda_0 > 0$, and $a_{\alpha,\lambda} > 0$, such that*

$$\left|dL\left(X^\delta\right) r_{\alpha,\lambda}(g)\right| \leqslant \begin{cases} a_{\alpha,\lambda} |g|^{-(d+|\delta|-q\alpha)} e^{-(b(\operatorname{Re}\lambda)^{1/q}-c)|g|}, & d > q\alpha, \\ a_{\alpha,\lambda}(1 + |\log|g||) e^{-(b(\operatorname{Re}\lambda)^{1/q}-c)|g|}, & d = q\alpha, \\ a_{\alpha,\lambda} e^{-(b(\operatorname{Re}\lambda)^{1/q}-c)|g|}, & d < q\alpha \end{cases}$$

for each $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \lambda_0$.

A proof of these estimates is given in [19, pp. 238 and 245]. Our next aim is to understand the microlocal structure of the operators $\pi(r_{\alpha,\lambda})$ on the Oshima compactification $\widetilde{\mathbb{X}}$ of $\mathbb{X} = G/K$. Consider again the atlas $\{(\widetilde{W}_\gamma, \varphi_\gamma^{-1})\}_{\gamma \in I}$ of $\widetilde{\mathbb{X}}$ introduced in Section 4, and the local operators

$$A^\gamma_{r_{\alpha,\lambda}} u = \big[\pi(r_{\alpha,\lambda})_{|\widetilde{W}_\gamma}\big(u \circ \varphi_\gamma^{-1}\big)\big] \circ \varphi_\gamma, \tag{28}$$

where $u \in \mathrm{C}^\infty_{\mathrm{c}}(W_\gamma)$ and $W_\gamma = \varphi_\gamma^{-1}(\widetilde{W}_\gamma)$. By the Fourier inversion formula, $A^\gamma_{r_{\alpha,\lambda}}$ is given by the absolutely convergent integral

$$A^\gamma_{r_{\alpha,\lambda}} u(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} a^\gamma_{r_{\alpha,\lambda}}(x,\xi)\hat{u}(\xi) đ\xi, \tag{29}$$

where

$$a^\gamma_{r_{\alpha,\lambda}}(x,\xi) = \int_G e^{i(\varphi^g_\gamma(x)-x)\cdot\xi} c_\gamma(x,g) r_{\alpha,\lambda}(g) d_G(g),$$

$$\tilde{a}^\gamma_{r_{\alpha,\lambda}}(x,\xi) = \int_G e^{i\Psi_\gamma(g,x)\cdot\xi} c_\gamma(x,g) r_{\alpha,\lambda}(g) d_G(g)$$

are smooth functions on $W_\gamma \times \mathbb{R}^{k+l}$, since $r_{\alpha,\lambda} \in \mathrm{L}^1(G, e^{\kappa|g|}d_G)$. Moreover, in view of the L^1-bound (25), the functions $e^{-\lambda\tau}\tau^{\alpha-1}\tilde{a}^\gamma_{f_\tau}(x,\xi)$ and $e^{-\lambda\tau}\tau^{\alpha-1}a^\gamma_{f_\tau}(x,\xi)$ are integrable in τ over $(0,\infty)$, and by Fubini we obtain the equalities

$$a^\gamma_{r_{\alpha,\lambda}}(x,\xi) = \Gamma(\alpha)^{-1}\int_0^\infty e^{-\lambda\tau}\tau^{\alpha-1} a^\gamma_{f_\tau}(x,\xi) d\tau,$$

$$\tilde{a}^\gamma_{r_{\alpha,\lambda}}(x,\xi) = \Gamma(\alpha)^{-1}\int_0^\infty e^{-\lambda\tau}\tau^{\alpha-1} \tilde{a}^\gamma_{f_\tau}(x,\xi) d\tau.$$

In what follows, we shall describe the microlocal structure of the resolvent $(\lambda\mathbf{1} + \overline{\Omega})^{-\alpha}$ on $\widetilde{\mathbb{X}}$, and in particular, its kernel.

Proposition 3. *Let Q be the largest integer such that $Q < q\alpha$. Then $\tilde{a}^\gamma_{r_{\alpha,\lambda}}(x,\xi) \in \mathrm{S}^{-Q}_{la}(W_\gamma \times \mathbb{R}^{k+l})$. That is, for any compactum $\mathcal{K} \subset W_\gamma$, and arbitrary multi-indices β, ε there exist constants $C_{\mathcal{K},\beta,\varepsilon} > 0$ such that*

$$\big|\big(\partial^\varepsilon_x \partial^\beta_\xi \tilde{a}^\gamma_{r_{\alpha,\lambda}}\big)(x,\xi)\big| \leqslant C_{\mathcal{K},\beta,\varepsilon}\big(1+|\xi|^2\big)^{(-Q-|\beta|)/2}, \quad x \in \mathcal{K},\ \xi \in \mathbb{R}^{k+l}, \tag{30}$$

and $\tilde{a}^\gamma_{r_{\alpha,\lambda}}$ satisfies the lacunary condition (8) for each of the coordinates x_j, $k+1 \leqslant j \leqslant k+l$.

Proof. For a fixed chart $(\widetilde{W}_\gamma, \varphi_\gamma)$ of $\widetilde{\mathbb{X}}$ we write $x = (n,t) \in W_\gamma$, $\tilde{x} = \varphi_\gamma(x) \in \widetilde{W}_\gamma$, as usual. As a consequence of Proposition 2 and Lemma 4 one computes with (22) for arbitrary $N \in \mathbb{N}$

$$\begin{aligned}
\big(\partial_\xi^{2\beta}\tilde a^\gamma_{r_{\alpha,\lambda}}\big)(x,\xi) &= \int_G e^{i\Psi_\gamma(g,x)\cdot\xi}\big[i\Psi_\gamma(g,x)\big]^{2\beta} c_\gamma(x,g) r_{\alpha,\lambda}(g) d_G(g)\\
&= \big(1+|\xi|^2\big)^{-N} e^{-i(x_1,\dots,x_k,1,\dots,1)\cdot\xi}\sum_{r=0}^{2N}\sum_{|\delta|=r}\int_G b^N_\delta\big(x,g^{-1}\big) dL\big(X^\delta\big)\psi^\gamma_{\xi,x}\big(g^{-1}\big)\\
&\quad\cdot\big[i\Psi_\gamma(x,g)\big]^{2\beta} c_\gamma(x,g) r_{\alpha,\lambda}(g) d_G(g).
\end{aligned}$$

Now, $n_r(g\cdot\tilde x)\to n_r(\tilde x)$ and $\chi_r(g,\tilde x)\to 1$ as $g\to e$, so that due to the analyticity of the G-action on $\widetilde{\mathbb X}$ one deduces

$$\big|\Psi_\gamma(g,x)\big| = \big|\big(n_1\big(g^{-1}\cdot\tilde x\big)-n_1(\tilde x),\dots,\chi_1\big(g^{-1}\cdot\tilde x\big)-1,\dots\big)\big| \leqslant C_{\mathcal K}|g|,\quad x\in\mathcal K, \tag{31}$$

for some constant $C_{\mathcal K}$. Indeed, let

$$(\zeta_1,\dots,\zeta_d)\mapsto e^{\zeta_1X_1+\dots+\zeta_dX_d}=g$$

be canonical coordinates of the first type near the identity $e\in G$. We then have the power expansions

$$\chi_r(g,\tilde x)-1=\sum_{\alpha,\beta,\gamma} c^r_{\alpha,\beta,\gamma} n^\alpha t^\beta\zeta^\gamma,\qquad n_r(g\cdot\tilde x)-n_r(\tilde x)=\sum_{\alpha,\beta,\gamma} d^r_{\alpha,\beta,\gamma} n^\alpha t^\beta\zeta^\gamma, \tag{32}$$

where $c^r_{\alpha,\beta,\gamma}, d^r_{\alpha,\beta,\gamma}=0$ if $|\gamma|=0$. Hence,

$$\big|n_r(g\cdot\tilde x)-n_r(\tilde x)\big|,\big|\chi_r(g,\tilde x)-1\big|\leqslant C_1|\zeta|\leqslant C_2|g|,$$

compare [19, pp. 12–13], and we obtain (31). With Theorem 5, and, say, $d>q\alpha$, we therefore have the pointwise estimates

$$\big|\Psi_\gamma(g,x)^{\beta'} dL\big(X^{\delta'}\big) r_{\alpha,\lambda}(g)\big|\leqslant C_{\mathcal K,\alpha,\lambda}|g|^{-(d+|\delta'|-q\alpha-|\beta'|)} e^{-(b(\operatorname{Re}\lambda)^{1/q}-c)|g|}$$

for some constant $C_{\mathcal K,\alpha,\lambda}>0$ uniformly on $\mathcal K\times V_{\gamma,\tilde x}$. Now, let $2\tilde Q$ be the largest even number strictly smaller than $q\alpha$. Applying the same reasoning as in the proof of Proposition 1, one obtains for $N=\tilde Q+|\beta|$

$$\begin{aligned}
\big(\partial_\xi^{2\beta}\tilde a^\gamma_{r_{\alpha,\lambda}}\big)(x,\xi) &= \big(1+|\xi|^2\big)^{-\tilde Q-|\beta|}\sum_{r=0}^{2\tilde Q+2|\beta|}\sum_{|\delta|=r}(-1)^{|\delta|}\int_G e^{i\Psi_\gamma(g^{-1},x)\cdot\xi}\\
&\quad\cdot dL\big(X^{\tilde\delta}\big)\big[b_\delta^{\tilde Q+|\beta|}(x,g)\big[i\Psi_\gamma\big(g^{-1},x\big)\big]^{2\beta} c_\gamma\big(x,g^{-1}\big) r_{\alpha,\lambda}\big(g^{-1}\big)\big] d_G(g),
\end{aligned}$$

since all the occurring combinations $\Psi_\gamma(g^{-1},x)^{\beta'} dL(X^{\delta'})[r_{\alpha,\lambda}(g^{-1})]$ on the right hand side are such that $q\alpha+|\beta'|-|\delta'|>0$, implying that the corresponding integrals over G converge. Equality then follows by the left-invariance of $d_G(g)$, and Lebesgue's theorem on dominated convergence. To show the estimate (30) in general for $\varepsilon = 0$, let $x \in \mathcal{K}$, and $\xi \in \mathbb{R}^{k+l}$ be such that $|\xi| \geqslant 1$, and $|\xi|_{\max} = \max\{|\xi_r| : 1 \leqslant r \leqslant k+l\} = |\xi_j|$. Using (19) and (20) we can express $\xi_j^{Q+|\beta|}\psi^\gamma_{\xi,x}(g)$ as left derivatives of $\psi^\gamma_{\xi,x}(g)$, and repeating the previous argument we obtain the estimate

$$\begin{aligned}\left|\left(\partial^\beta_\xi \tilde{a}^\gamma_{r_{\alpha,\lambda}}\right)(x,\xi)\right| &= |\xi_j|^{-Q-|\beta|}\left|\sum_{r=0}^{Q+|\beta|}\sum_{|\delta|=r}\int_G b^j_\delta\left(x,g^{-1}\right)dL\left(X^\delta\right)\psi^\gamma_{\xi,x}\left(g^{-1}\right)\right. \\ &\qquad \left.\cdot\left[i\varPsi_\gamma(x,g)\right]^\beta c_\gamma(x,g)r_{\alpha,\lambda}(g)d_G(g)\right| \\ &\leqslant \tilde{C}_{\mathcal{K},\beta}\frac{1}{|\xi|_{\max}^{Q+|\beta|}} \leqslant C_{\mathcal{K},\beta}\frac{1}{|\xi|^{Q+|\beta|}}, \end{aligned} \tag{33}$$

where the coefficients $b^j_\delta(x,g)$ are at most of exponential growth in g. But since $\tilde{a}^\gamma_{r_{\alpha,\lambda}}(x,\xi) \in \mathrm{C}^\infty(W_\gamma \times \mathbb{R}^{k+l})$, we obtain (30) for $\varepsilon = 0$. Let us now turn to the x-derivatives. We have to show that the powers in ξ that arise when differentiating $(\partial^\beta_\xi \tilde{a}^\gamma_{r_{\alpha,\lambda}})(x,\xi)$ with respect to x can be compensated by an argument similar to the previous considerations. Now, (32) clearly implies

$$\partial^\varepsilon_x\left(\chi_r(g,\tilde{x})-1\right) = O\left(|g|\right), \qquad \partial^\varepsilon_x\left(n_r(g\cdot\tilde{x})-n_r(\tilde{x})\right) = O\left(|g|\right).$$

Thus, each time we differentiate the exponential $e^{i\varPsi_\gamma(g,x)\cdot\xi}$ with respect to x, the result is of order $O(|\xi||g|)$. Therefore, expressing the occurring powers $\xi^{\varepsilon'}\psi^\gamma_{\xi,x}(g)$ as left derivatives of $\psi^\gamma_{\xi,x}(g)$, we can repeat the preceding argument to absorb the powers in ξ, and (30) follows. Note next that the previous argument also implies $a^\gamma_{r_{\alpha,\lambda}}(x,\xi) \in \mathrm{S}^{-Q}(W^*_\gamma \times \mathbb{R}^{k+l}_\xi)$, where we wrote $W^*_\gamma = \{x = (n,t) \in W_\gamma : t_1 \cdots t_l \neq 0\}$, the G-action being transitive on each $\widetilde{\mathbb{X}}_\Delta$. The Schwartz kernel $K_{A^\gamma_{r_{\alpha,\lambda}}}$ of the restriction of the operator (28) to W^*_γ is therefore given by the oscillatory integral

$$\int e^{i(x-y)\cdot\xi}a^\gamma_{r_{\alpha,\lambda}}(x,\xi)\,đ\xi \in \mathcal{D}'\left(W^*_\gamma \times W^*_\gamma\right),$$

which is C^∞ off the diagonal. As in (23) we have $\operatorname{supp} K_{A^\gamma_{r_{\alpha,\lambda}}} \subset \bigcup_{\Theta\subset\Delta}\overline{W^\Theta_\gamma}\times\overline{W^\Theta_\gamma}$, so that each of the integrals

$$\int e^{i(x_j-y_j)\xi_j}\tilde{a}^\gamma_{r_{\alpha,\lambda}}\left(x,(\mathbf{1}_k\otimes T_x)\xi\right)d\xi_j, \quad j = k+1,\ldots,k+l,$$

must vanish if x_j and y_j do not have the same sign. Hence,

$$\int e^{-ir_j\xi_j}\tilde{a}^\gamma_{r_{\alpha,\lambda}}(x,\xi)d\xi_j = 0 \quad \text{for } r_j < -1,\ x \in W^*_\gamma.$$

Since $\tilde{a}^{\gamma}_{r_{\alpha,\lambda}}(x,\xi) \in \mathrm{S}^{-Q}(W_\gamma \times \mathbb{R}^{k+l}_\xi)$, these integrals are absolutely convergent for $r_j \neq 0$. Lebesgue's theorem on bounded convergence theorem then implies that these conditions must also hold for $x \in W_\gamma$. The proof of the proposition is now complete. $\square$

Remark 2. One would actually expect that $\tilde{a}^{\gamma}_{r_{\alpha,\lambda}}(x,\xi) \in \mathrm{S}^{-q\alpha}_{la}(W_\gamma \times \mathbb{R}^{k+l})$, being the local symbol of the resolvent $(\lambda\mathbf{1}+\overline{\Omega})^{-\alpha}$. Nevertheless, the general estimates of Theorem 5 for the resolvent kernels $r_{\alpha,\lambda}$, which correctly reflect the singular behavior at the identity, are not sufficient to show this, and more information about them is required. Indeed, $dL(X^\beta)r_{\alpha,\lambda} \in L_1(G, d_G(g))$ only holds if $0 < q\alpha - |\beta|$.

We are now able to describe the microlocal structure of the resolvent $(\lambda\mathbf{1}+\overline{\Omega})^{-\alpha}$.

Theorem 6. *Let Ω be a strongly elliptic differential operator of order q associated with the representation $(\pi, C(\widetilde{\mathbb{X}}))$ of G. Let $\omega \geqslant 0$ be given by Theorem 3, and $\lambda \in \mathbb{C}$ be such that $\operatorname{Re}\lambda > \omega$. Let further $\alpha > 0$, and denote by Q the largest integer such that $Q < q\alpha$. Then $(\lambda\mathbf{1}+\overline{\Omega})^{-\alpha} = \pi(r_{\alpha,\lambda})$ is locally of the form (29), where $a^{\gamma}_{r_{\alpha,\lambda}}(x,\xi) = \tilde{a}^{\gamma}_{r_{\alpha,\lambda}}(x,(\mathbf{1}_k \otimes T_x)\xi)$, and $\tilde{a}^{\gamma}_{r_{\alpha,\lambda}}(x,\xi) \in \mathrm{S}^{-Q}_{la}(W_\gamma \times \mathbb{R}^{k+l})$. In particular, $(\lambda\mathbf{1}+\overline{\Omega})^{-\alpha}$ is a totally characteristic pseudodifferential operators of class L^{-Q}_b on the manifolds with corners $\overline{\widetilde{\mathbb{X}}_\Delta}$. Furthermore, its kernel is locally given by the oscillatory integral*

$$R^{\gamma}_{\alpha,\lambda}(x,y) = \int e^{i(x-y)\xi} a^{\gamma}_{r_{\alpha,\lambda}}(x,\xi)\,đ\xi = \frac{1}{|x_{k+1}\cdots x_{k+l}|}\int e^{i(\mathbf{1}_k\otimes T_x^{-1})(x-y)\cdot\xi}\tilde{a}^{\gamma}_{r_{\alpha,\lambda}}(x,\xi)\,đ\xi,$$

where $x_{k+1}\cdots x_{k+l} \neq 0$, $x, y \in W_\gamma$. $R^{\gamma}_{\alpha,\lambda}(x,y)$ is smooth off the diagonal, and rapidly falling as $|x_j| \to 0$, or $|y_j| \to 0$, as long as $x_j \neq y_j$, where $k+1 \leqslant j \leqslant k+l$.

Proof. The assertions of the theorem are direct consequences of our previous considerations, except for the behavior of $R^{\gamma}_{\alpha,\lambda}(x,y)$ at infinity. Let $k+1 \leqslant j \leqslant k+l$. While the behavior as $|y_j| \to 0$ is a direct consequence of the lacunarity of $\tilde{a}^{\gamma}_{R_{\alpha,\gamma}}$, the behavior as $|x_j| \to 0$ is a direct consequence of the fact that, as oscillatory integrals,

$$\int e^{i(\mathbf{1}_k\otimes T_x^{-1})(x-y)\cdot\xi}\tilde{a}^{\gamma}_{r_{\alpha,\lambda}}(x,\xi)\,đ\xi = \frac{1}{|(\mathbf{1}_k \otimes T_x^{-1})(x-y)|^{2N}}\int e^{i(x-y)\cdot\xi}\Delta^N_\xi \tilde{a}^{\gamma}_{r_{\alpha,\lambda}}(x,\xi)\,đ\xi,$$

where $\Delta_\xi = \partial^2_{\xi_1} + \cdots + \partial^2_{\xi_{k+l}}$, $x \neq y$, and N is arbitrarily large. $\square$

Remark 3. The singular behavior of $r_{\alpha,\lambda}(g)$ at the identity corresponds to the fact that, as a pseudodifferential operator of class L^{-Q}_b, $(\lambda\mathbf{1}+\overline{\Omega})^{-\alpha}$ has a kernel which is singular at the diagonal.

To conclude, let us say some words about the classical heat kernel on a Riemannian symmetric space of non-compact type. Consider thus the regular representation $(\sigma, \mathrm{C}(\widetilde{\mathbb{X}}))$

of the solvable Lie group $S = AN^- \simeq \mathbb{X} = G/K$ on the Oshima compactification $\widetilde{\mathbb{X}}$ of $\mathbb{X}$, and associate to every $f \in \mathcal{S}(S)$ the corresponding convolution operator

$$\int_S f(g)\sigma(g)d_S(g).$$

Its restriction to $\mathrm{C}^\infty(\widetilde{\mathbb{X}})$ induces again a continuous linear operator

$$\sigma(f) : \mathrm{C}^\infty(\widetilde{\mathbb{X}}) \to \mathrm{C}^\infty(\widetilde{\mathbb{X}}) \subset \mathcal{D}'(\widetilde{\mathbb{X}}),$$

and an examination of the arguments in Section 7 shows that an analogous analysis applies to the operators $\sigma(f)$. In particular, Theorem 2 holds for them, too. Let ϱ be the half sum of all positive roots, and

$$C = \sum_j H_j^2 - \sum_j Z_j^2 - \sum_j \big[X_j\theta(X_j) + \theta(X_j)X_j\big] \equiv \sum_j H_j^2 - 2\varrho + 2\sum X_j^2 \mod \mathfrak{U}(\mathfrak{g})\mathfrak{k}$$

be the Casimir operator in $\mathfrak{U}(\mathfrak{g})$, where $\{H_j\}$, $\{Z_j\}$, and $\{X_j\}$ are orthonormal basis of $\mathfrak{a}$, $\mathfrak{m}$, and $\mathfrak{n}^-$, respectively, and put $C' = \sum_j H_j^2 - 2\varrho + 2\sum X_j^2$. Though $-d\pi(C')$ is not a strongly elliptic operator in the sense defined above, $\Omega = -d\sigma(C')$ certainly is. Consequently, if $f'_\tau(g) \in \mathcal{S}(S)$ denotes the corresponding Langlands kernel, Theorems 4 and 6 yield descriptions of the Schwartz kernels of $\sigma(f'_\tau)$ and $(\lambda\mathbf{1} + \overline{\Omega})^{-\alpha}$ on $\widetilde{\mathbb{X}}$. On the other hand, denote by Δ the Laplace–Beltrami operator on $\mathbb{X}$. Then

$$\Delta\varphi(gK) = \varphi(g : C) = \varphi\big(g : C'\big), \quad \varphi \in \mathrm{C}^\infty(\mathbb{X}),$$

and the associated heat kernel $h_\tau(g)$ on $\mathbb{X}$ coincides with the heat kernel on S associated to C'. But the latter is essentially given by the Langlands kernel $f'_\tau(g)$, being the solution of the parabolic equation

$$\frac{\partial f'_\tau}{\partial \tau}(g) - dL\big(C'\big)f'_\tau(g) = 0, \qquad \lim_{\tau\to 0} f'_\tau(g) = \delta(g)$$

on S. In this particular case, optimal upper and lower bounds for h_τ and the Bessel–Green–Riesz kernels were given in [1] using spherical analysis under certain restrictions coming from the lack of control in the Trombi–Varadarajan expansion for spherical functions along the walls. Our asymptotics for the kernels of $\sigma(f'_\tau)$ and $(\lambda\mathbf{1} + \overline{\Omega})^{-\alpha}$ on $\widetilde{\mathbb{X}}_\Delta \simeq \mathbb{X}$ are free of restrictions, and in concordance with those of [1], though, of course, less explicit. A detailed description of the resolvent of Δ on $\mathbb{X}$ was given in [12,14].

6. Regularized traces

We shall now define a regularized trace for the convolution operators $\pi(f)$ introduced in Section 4. To begin with, recall that, as a consequence of Theorem 2, we can write the kernel of $\pi(f)$ locally in the form

$$\begin{aligned} K_{A_f^\gamma}(x,y) &= \int e^{i(x-y)\cdot\xi} a_f^\gamma(x,\xi) d\!\!\!{}^-\xi = \int e^{i(x-y)\cdot(\mathbf{1}_k \otimes T_x^{-1})\xi} \tilde{a}_f^\gamma(x,\xi) \left|\det\left(\mathbf{1}_k \otimes T_x^{-1}\right)'(\xi)\right| d\!\!\!{}^-\xi \\ &= \frac{1}{|x_{k+1}\cdots x_{k+l}|} \tilde{A}_f^\gamma\left(x, x_1-y_1, \dots, 1-\frac{y_{k+1}}{x_{k+1}}, \dots\right), \quad x_{k+1}\cdots x_{k+l} \neq 0, \end{aligned} \tag{34}$$

where $\tilde{A}_f^\gamma(x,y)$ denotes the inverse Fourier transform of the lacunary symbol $\tilde{a}_f^\gamma(x,\xi)$ given by (24). Consider now the partition of unity $\{\alpha_\gamma\}$ subordinate to the atlas $\{(\widetilde{W}_\gamma, \varphi_\gamma^{-1})\}$. By (34), the restriction of the kernel of A_f^γ to the diagonal is given by

$$K_{A_f^\gamma}(x,x) = \frac{1}{|x_{k+1}\cdots x_{k+l}|} \tilde{A}_f^\gamma(x,0), \quad x_{k+1}\cdots x_{k+l} \neq 0.$$

These restrictions yield a family of smooth functions $k_f^\gamma(\tilde{x}) = K_{A_f^\gamma}(\varphi_\gamma^{-1}(\tilde{x}), \varphi_\gamma^{-1}(\tilde{x}))$ which define a density k_f on

$$2^{\#l}(G/K) \subset \widetilde{\mathbb{X}}.$$

Nevertheless, the functions $k_f^\gamma(\tilde{x})$ are not locally integrable on the entire compactification $\widetilde{\mathbb{X}}$, so that we cannot define a trace of $\pi(f)$ by integrating the density k_f over the diagonal $\Delta_{\widetilde{\mathbb{X}}\times\widetilde{\mathbb{X}}} \simeq \widetilde{\mathbb{X}}$ as in (7). Instead, we have the following

Proposition 4. *Let $f \in \mathcal{S}(G)$, $s \in \mathbb{C}$, and define for* $\operatorname{Re} s > 0$

$$\begin{aligned} \operatorname{Tr}_s \pi(f) &= \sum_\gamma \int_{W_\gamma} (\alpha_\gamma \circ \varphi_\gamma)(x) |x_{k+1}\cdots x_{k+l}|^s \widetilde{A}_f^\gamma(x,0)\, dx \\ &= \left\langle |x_{k+1}\cdots x_{k+l}|^s, \sum_\gamma (\alpha_\gamma \circ \varphi_\gamma) \widetilde{A}_f^\gamma(\cdot,0) \right\rangle. \end{aligned}$$

Then $\operatorname{Tr}_s \pi(f)$ can be continued analytically to a meromorphic function in s with at most poles at $-1, -3, \dots$. Furthermore, for $s \in \mathbb{C} - \{-1,-3,\dots\}$,

$$\Theta_\pi^s : \mathrm{C}_\mathrm{c}^\infty(G) \ni f \mapsto \operatorname{Tr}_s \pi(f) \in \mathbb{C} \tag{35}$$

defines a distribution density on G.

Proof. The fact that $\operatorname{Tr}_s \pi(f)$ can be continued meromorphically is a consequence of the analytic continuation of $|x_{k+1}\cdots x_{k+l}|^s$ as a distribution in $\mathbb{R}^{k+l}$, proved by Bernshtein–Gel'fand in [5, Lemma 2]. One even has that

$$\left\langle |x_{k+1}|^{s_1} \cdots |x_{k+l}|^{s_l}, u \right\rangle, \quad u \in \mathrm{C}_\mathrm{c}^\infty\left(\mathbb{R}^{k+l}\right),$$

can be continued meromorphically in the variables $s_1, \dots, s_l$ to $\mathbb{C}^l$ with poles $s_i = -1, -3, \dots$. To see that (35) is a distribution density, note that $\Theta^s_\pi : \mathrm{C}^\infty_{\mathrm{c}}(G) \to \mathbb{C}$ is certainly linear. Since $|x_{k+1} \cdots x_{k+l}|^s$ is a distribution, for any open, relatively compact subset $\omega \subset \mathbb{R}^{k+l}$ there exist $C_\omega > 0$ and $B_\omega \in \mathbb{N}$ such that

$$\left|\left\langle |x_{k+1} \cdots x_{k+l}|^s, u\right\rangle\right| \leqslant C_\omega \sum_{|\beta| \leqslant B_\omega} \sup\left|\partial^\beta u\right|, \quad u \in \mathrm{C}^\infty_{\mathrm{c}}(\omega). \tag{36}$$

Let now $\mathcal{O} \subset G$ be an arbitrary open, relatively compact subset, and $f \in \mathrm{C}^\infty_{\mathrm{c}}(\mathcal{O})$. With (24) one has

$$\operatorname{Tr}_s \pi(f) = \left\langle |x_{k+1} \cdots x_{k+l}|^s, \sum_\gamma (\alpha_\gamma \circ \varphi_\gamma) \int \tilde{a}^\gamma_f(\cdot, \xi) đ\xi \right\rangle. \tag{37}$$

By (22), one computes for arbitrary $N \in \mathbb{N}$ that

$$e^{i\Psi_\gamma(g,x)\cdot\xi} = \frac{1}{(1+|\xi|^2)^N} \sum_{r=0}^{2N} \sum_{|\alpha|=r} b^N_\alpha\left(x, g^{-1}\right) dL\left(X^\alpha\right)\left[e^{i\Psi_\gamma(\cdot^{-1},x)\cdot\xi}\right]\left(g^{-1}\right),$$

where the coefficients $b^N_\alpha(x,g)$ are smooth, and at most of exponential growth in g. With (14) and Proposition 1 we therefore obtain for $\tilde{a}^\gamma_f(x,\xi)$ the expression

$$\begin{aligned}
&\tilde{a}^\gamma_f(x,\xi) \\
&= \frac{1}{(1+|\xi|^2)^N} \int_G e^{i\Psi_\gamma(g^{-1},x)\cdot\xi} \sum_{r=0}^{2N} \sum_{|\alpha|=r} (-1)^r dL\left(X^{\tilde{\alpha}}\right)\left[b^N_\alpha(x,g) c_\gamma\left(x, g^{-1}\right) f\left(g^{-1}\right)\right] d_G(g).
\end{aligned}$$

Inserting this in (37), and taking N sufficiently large, we obtain with (36) that

$$\left|\operatorname{Tr}_s \pi(f)\right| \leqslant C_{\mathcal{O}} \sum_{|\beta| \leqslant B_{\mathcal{O}}} \sup\left|dL\left(X^\beta\right) f\right|$$

for suitable $C_{\mathcal{O}} > 0$ and $B_{\mathcal{O}} \in \mathbb{N}$. Since the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}_\mathbb{C})$ can be identified with the algebra of invariant differential operators on G, the assertion now follows with [22, p. 480]. $\square$

Remark 4. Using Hironaka's theorem on resolution of singularities, Bernstein–Gel'fand [5] and Atiyah [2] even proved the following general result. Let M be a real analytic manifold and f a non-zero, real analytic function on M. Then $|f|^s$, which is locally integrable for $\operatorname{Re} s > 0$, extends analytically to a distribution on M which is a meromorphic function of s in the whole complex plane. The poles are located at the negative rational numbers, and their order does not exceed the dimension of M. From this one deduces that if $f : M \to \mathbb{C}$ is a non-zero analytic function, then there exists a distribution

S on M such that $fS = 1$. This is the Hörmander–Lojasiewicz theorem on the division of distributions, and implies the existence of temperate fundamental solutions for constant-coefficient differential operators.

Consider next the Laurent expansion of $\Theta_\pi^s(f)$ at $s = -1$. For this, let $u \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^{k+l})$ be a test function, and consider the expansion

$$\left\langle |x_{k+1} \cdots x_{k+l}|^s, u \right\rangle = \sum_{j=-q}^{\infty} S_j(u)(s+1)^j,$$

where $S_k \in \mathcal{D}'(\mathbb{R}^{k+l})$. Since $|x_{k+1} \cdots x_{k+l}|^{s+1}$ has no pole at $s = -1$, we necessarily must have

$$|x_{k+1} \cdots x_{k+l}| \cdot S_j = 0 \quad \text{for } j < 0, \qquad |x_{k+1} \cdots x_{k+l}| \cdot S_0 = 1$$

as distributions. Therefore $S_0 \in \mathcal{D}'(\mathbb{R}^{k+l})$ represents a distributional inverse of $|x_{k+1} \cdots x_{k+l}|$. By repeating the reasoning of the proof of Proposition 4 we arrive at the following

Proposition 5. *For $f \in \mathcal{S}(G)$, let the regularized trace of the operator $\pi(f)$ be defined by*

$$\mathrm{Tr}_{reg}\, \pi(f) = \left\langle S_0, \sum_\gamma (\alpha_\gamma \circ \varphi_\gamma) \tilde{A}_f^\gamma(\cdot, 0) \right\rangle.$$

Then $\Theta_\pi : \mathrm{C}_\mathrm{c}^\infty(G) \ni f \mapsto \mathrm{Tr}_{reg}\, \pi(f) \in \mathbb{C}$ constitutes a distribution density on G, which is called the character of the representation π.

Remark 5. An alternative definition of $\mathrm{Tr}_{reg}\, \pi(f)$ could be given within the calculus of b-pseudodifferential operators developed by Melrose. For a detailed description, the reader is referred to [11, Section 6].

In what follows, we shall identify distributions with distribution densities on G via the Haar measure d_G. Our next aim is to understand the distributions Θ_π^s and Θ_π in terms of the G-action on $\widetilde{\mathbb{X}}$. We shall actually show that on a certain open set of transversal elements, they are represented by locally integrable functions given in terms of fixed points. For this, we shall first review some largely known facts about group actions on homogeneous spaces.

7. Fixed points of group actions on homogeneous spaces

Let G be a Lie group with Lie algebra $\mathfrak{g}$, $H \subset G$ a closed subgroup with Lie algebra $\mathfrak{h}$, and $\pi : G \to G/H$ the canonical projection. For an element $g \in G$, consider the natural left action $l_g : G/H \to G/H$ given by $l_g(xH) = gxH$. Let Ad^G denote the adjoint action of G on $\mathfrak{g}$.

Lemma 5.

(1) *$l_{g^{-1}} : G/H \to G/H$ has a fixed point if and only if $g \in \bigcup_{x\in G} xHx^{-1}$. Moreover, to every fixed point xH one can associate a unique conjugacy class $h(g, xH)$ in H.*
(2) *Let xH be a fixed point of $l_{g^{-1}}$ and let $h \in h(g, xH)$. Then*

$$\det(\mathbf{1} - dl_{g^{-1}})_{xH} = \det\big(\mathbf{1} - \mathrm{Ad}^G_H(h)\big),$$

where $\mathrm{Ad}^G_H : H \to \mathrm{Aut}(\mathfrak{g}/\mathfrak{h})$ is the isotropy action of H on $\mathfrak{g}/\mathfrak{h}$.

Proof. See e.g. [4, p. 463]. □

Consider now the case when G is a connected, real, semi-simple Lie group with finite center, θ a Cartan involution of $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan decomposition. Further, let K be the maximal compact subgroup of G associated to $\mathfrak{k}$, and consider the corresponding Riemannian symmetric space $\mathbb{X} = G/K$ which is assumed to be of non-compact type. By definition, θ is an involutive automorphism of $\mathfrak{g}$ such that the bilinear form $\langle\cdot,\cdot\rangle_\theta$ is strictly positive definite. In particular, $\langle\cdot,\cdot\rangle_{\theta|\mathfrak{p}\times\mathfrak{p}}$ is a symmetric, positive-definite, bilinear form, yielding a left-invariant metric on G/K. Endowed with this metric, G/K becomes a complete, simply connected, Riemannian manifold with non-positive sectional curvature. Such manifolds are called *Hadamard manifolds*. Furthermore, for each $g \in G$, $l_{g^{-1}} : G/K \to G/K$ is an isometry on G/K with respect to this left-invariant metric. Note that Riemannian symmetric spaces of non-compact type are precisely the simply connected Riemannian symmetric spaces with sectional curvature $\kappa \leqslant 0$ and with no Euclidean de Rham factor.

Next, let M be a smooth manifold, and recall that a fixed point x_0 of a smooth map $f : M \to M$ is said to be *simple* if $\det(\mathbf{1} - df_{x_0}) \neq 0$, where df_{x_0} denotes the differential of f at x_0. The map f is called *transversal* if it has only simple fixed points. Note that the non-vanishing condition on the determinant is equivalent to the requirement that the graph of f intersects the diagonal transversally at $(x_0, x_0) \in M \times M$, and hence the terminology. In particular, a simple fixed point is an isolated fixed point. We then have the following

Lemma 6. *Let $g \in G$ be such that $l_{g^{-1}} : G/K \to G/K$ is transversal. Then $l_{g^{-1}}$ has a unique fixed point in G/K.*

Proof. Let M be a Hadamard manifold, and φ an isometry on M that leaves two distinct points $x, y \in M$ fixed. By general theory, there is a unique minimal geodesic $\gamma : \mathbb{R} \to M$ joining x and y. Let $\gamma(0) = x$ and $\gamma(1) = y$, so that $\varphi \circ \gamma(0) = \varphi(x) = x$ and $\varphi \circ \gamma(1) = \varphi(y) = y$. Since isometries take geodesics to geodesics, $\varphi \circ \gamma$ is a geodesic in M, joining x and y. By the uniqueness of γ we therefore conclude that $\varphi \circ \gamma = \gamma$. This means that an isometry on a Hadamard manifold with two distinct fixed points also fixes the unique geodesic joining them point by point. Since, by assumption, $l_{g^{-1}} : G/K \to G/K$ has only isolated fixed points, the lemma follows. □

In what follows, we shall call an element $g \in G$ *transversal relative to a closed subgroup* H if $l_{g^{-1}} : G/H \to G/H$ is transversal, and denote the set of all such elements by $G(H)$.

Proposition 6. *Let G be a connected, real, semi-simple Lie group with finite center, and K a maximal compact subgroup of G. Suppose* $\operatorname{rank}(G) = \operatorname{rank}(K)$. *Then any regular element of G is transversal relative to K. In other words, $G' \subset G(K)$, where G' denotes the set of regular elements in G.*

Proof. If a regular element g is such that $l_{g^{-1}} : G/K \to G/K$ has no fixed points, it is of course transversal. Let, therefore, $g \in G'$ be such that $l_{g^{-1}}$ has a fixed point x_0K. By Lemma 5, g must be conjugate to an element $k(g, x_0)$ in K. Consider now a maximal family of mutually non-conjugate Cartan subgroups $J_1, \dots, J_r$ in G, and put $J_i' = J_i \cap G'$ for $i \in \{1, \dots, r\}$. A result of Harish-Chandra then implies that

$$G' = \bigcup_{i=1}^{r} \bigcup_{x \in G} x J_i' x^{-1},$$

see [22, Theorem 1.4.1.7]. From this we deduce that

$$g = x k(g, x_0) x^{-1} = y j y^{-1} \quad \text{for some } x, y \in G,\ j \in J_i' \text{ for some } i.$$

Hence, $k(g, x_0)$ must be regular. Now, let T be a maximal torus of K. It is a Cartan subgroup of K, and the assumption that $\operatorname{rank}(G) = \operatorname{rank}(K)$ implies that T is also Cartan in G. Let $k(g, x_0K)$ be the conjugacy class in K associated to x_0K, as in Lemma 5. As K is compact, the maximal torus T intersects every conjugacy class in K. Varying x_0 over the coset x_0K, we can therefore assume that $k(g, x_0) \in k(g, x_0K) \cap T$. Thus, we conclude that $k(g, x_0) \in T \cap G'$. Note that, in particular, we can choose $J_i = T$ by the maximality of the family $J_1, \dots, J_r$. Now, for a regular element $h \in G$ belonging to a Cartan subgroup H one necessarily has $\det(\mathbf{1} - \operatorname{Ad}_H^G(h)) \neq 0$, compare the proof of Proposition 1.4.2.3 in [22]. Therefore $\det(\mathbf{1} - \operatorname{Ad}_T^G(k(g, x_0))) \neq 0$, and consequently, $\det(\mathbf{1} - \operatorname{Ad}_K^G(k(g, x_0))) \neq 0$. The assertion of the proposition now follows from Lemma 5. □

Corollary 3. *Let G be a connected, real, semi-simple Lie group with finite center, K a maximal compact subgroup of G, and suppose that* $\operatorname{rank}(G) = \operatorname{rank}(K)$. *Then the set of transversal elements $G(K)$ is dense in G.*

Proof. Since the set of regular elements G' is dense in G, the corollary follows from the previous proposition. □

Remark 6. Let us remark that with G as above, and P a parabolic subgroup of G, it is a classical result that $G' \subset G(P)$, see [8, p. 51].

8. Transversal trace and character formulae

We are now ready to describe the distributions Θ_π^s and Θ_s as locally integrable functions in terms of the fixed points of the G-action on $\widetilde{\mathbb{X}}$. Similar expressions were derived by Atiyah and Bott for the global character of an induced representation of G in [3], where they extended the classical Lefschetz fixed point theorem to geometric endomorphisms on elliptic complexes. Their work relies on the concept of transversal trace of a smooth operator, and its extension by continuity to pseudodifferential operators, which we now recall.

Let U be an open subset of $\mathbb{R}^n$, V open in U, and consider a smooth map $\alpha : V \to U$ with a simple fixed point at x_0. We choose V so small, that $x \mapsto x - \alpha(x)$ defines a diffeomorphism of V onto its image. Let $\Lambda : V \to U \times U$ be the map $\Lambda(x) = (\alpha(x), x)$, and assume that $A \in \mathrm{L}^{-\infty}(U)$ is a smooth operator with symbol $a(x,\xi)$. The kernel K_A of A is a smooth function on $U \times U$, and its restriction $\Lambda^* K_A$ to the graph of α defines a distribution on V according to

$$\begin{aligned}\langle \Lambda^* K_A, v\rangle &= \iint e^{i(\alpha(x)-x)\cdot\xi} a\bigl(\alpha(x),\xi\bigr) v(x)\,đ\xi\, dx \\ &= \iint e^{-iy\cdot\xi} \frac{a(\alpha(x(y)),\xi)v(x(y))}{|\det(\mathbf{1} - d\alpha(x(y)))|}\, dy\, đ\xi, \quad v \in \mathrm{C}_\mathrm{c}^\infty(V),\end{aligned} \tag{38}$$

where we made the substitution $y = x - \alpha(x)$, and the change in order of integration is permissible because $a(x,\xi) \in \mathrm{S}^{-\infty}(U)$. Now, for $a(x,\xi) \in \mathrm{S}^l(U)$, we observe that by differentiating

$$\int e^{-iy\cdot\xi} a\bigl(\alpha\bigl(x(y)\bigr),\xi\bigr) \frac{v(x(y))}{|\det(\mathbf{1} - d\alpha(x(y)))|}\, dy$$

with respect to ξ, and integrating by parts with respect to y, we obtain the estimate

$$\left| \partial_\xi^\gamma \int e^{-iy\cdot\xi} a\bigl(\alpha\bigl(x(y)\bigr),\xi\bigr) \frac{v(x(y))}{|\det(\mathbf{1} - d\alpha(x(y)))|}\, dy \right| \leqslant C\langle\xi\rangle^{l-|\beta|}$$

for arbitrary multi-indices γ and β and some constant $C > 0$. Thus, as an oscillatory integral, the last expression in (38) defines a distribution on V for any $a(x,\xi) \in \mathrm{S}^l(U)$. The distribution $\Lambda^* K_A$ is called *the transversal trace of* $A \in \mathrm{L}^l(U)$. If, in particular, $a(x,\xi) = a(x)$ is a polynomial of degree zero in ξ, one computes that

$$\Lambda^* K_A = \frac{a(x_0)\delta_{x_0}}{|\det(1 - d\alpha(x_0))|}. \tag{39}$$

This discussion can be globalized. Let $\mathbf{X}$ be a smooth manifold, E a vector bundle over $\mathbf{X}$, $\alpha : \mathbf{X} \to \mathbf{X}$ a C^∞-map with only simple fixed points, and $A : \Gamma_c(\alpha^* E) \to \Gamma(E)$ a pseudodifferential operator of order l between smooth sections. Denote the density bundle on $\mathbf{X}$ by Ω, put $F = \alpha^* E$, and define $F' = F^* \otimes \Omega$. The kernel K_A is then a distributional section of $E \boxtimes F'$. In other words, $K_A \in \mathcal{D}'(E \boxtimes F') = \mathcal{D}'(\mathbf{X} \times \mathbf{X}, E \boxtimes F')$. Similarly, one has $K_{\alpha^* A} \in \mathcal{D}'(\mathbf{X} \times \mathbf{X}, F \boxtimes F')$, where $\alpha^* A$ denotes the composition $\alpha^* A : \Gamma_c(F) \xrightarrow{A} \Gamma(E) \xrightarrow{\alpha^*} \Gamma(F)$. If $A \in \mathrm{L}^{-\infty}(F, E)$, K_A is a smooth section on $\mathbf{X} \times \mathbf{X}$, and $K_A(\tilde{x}, \tilde{y}) \in E_{\tilde{x}} \otimes F'_{\tilde{y}}$. In this case, $K_{\alpha^* A}(\tilde{x}, \tilde{y}) = K_A(\alpha(\tilde{x}), \tilde{y})$, so that one deduces $K_{\alpha^* A}(\tilde{x}, \tilde{x}) \in E_{\alpha(\tilde{x})} \otimes F'_{\tilde{x}} = F_{\tilde{x}} \otimes (F^* \otimes \Omega)_{\tilde{x}} \simeq \mathcal{L}(F_{\tilde{x}}, F_{\tilde{x}}) \otimes \Omega_{\tilde{x}}$. As a consequence, $\operatorname{Tr} K_{\alpha^* A}(\tilde{x}, \tilde{x})$ becomes a section of Ω, where Tr denotes the bundle homomorphism

$$\operatorname{Tr} : F \otimes F' \to \Omega. \tag{40}$$

Hence, if $\mathbf{X}$ is compact, one can define the trace of $\alpha^* A$ as

$$\operatorname{Tr} \alpha^* A = \int_{\mathbf{X}} \operatorname{Tr} K_{\alpha^* A}(\tilde{x}, \tilde{x}).$$

This trace can be extended to arbitrary $A \in \mathrm{L}^l(\mathbf{X})$. Indeed, for compact $\mathbf{X}$, the map $\mathrm{L}^{-\infty}(F, E) \to \mathbb{C}$, $A \to \operatorname{Tr} \alpha^* A$ has a unique continuous extension

$$\operatorname{Tr}_\alpha : \mathrm{L}^l(F, E) \to \mathbb{C}, \quad A \mapsto \operatorname{Tr}_\alpha A = \langle \operatorname{Tr} \Theta(A), 1 \rangle,$$

called the *transversal trace of* A, see [3, Proposition 5.3]. In the case that A is induced by a bundle homomorphism φ, it follows from (39) that

$$\operatorname{Tr}_\alpha A = \sum_{\tilde{x} \in \mathrm{Fix}(\alpha)} \nu_{\tilde{x}}(A), \quad \nu_{\tilde{x}}(A) = \frac{\operatorname{Tr} \varphi_{\tilde{x}}}{|\det(\mathbf{1} - d\alpha(\tilde{x}))|}, \tag{41}$$

the sum being over the fixed points of α on $\mathbf{X}$, see [3, Corollary 5.4].

In the context of representation theory, this trace was employed by Atiyah and Bott in [4] to compute the global character of an induced representation. To explain this, let G be a Lie group, H a closed subgroup of G, and ϱ a representation of H on a finite dimensional vector space V. The representation of G induced by ϱ is a geometric endomorphism in the space of sections over G/H with values in the homogeneous vector bundle $G \times_H V$, and shall be denoted by $T(g) = (\iota_* \varrho)(g)$. Assume that G/H is compact, and let d_G be a Haar measure on G. Consider a compactly supported smooth function $f \in \mathrm{C}^\infty_c(G)$, and the corresponding convolution operator $T(f) = \int_G f(g) T(g) d_G(g)$. It is a smooth operator, and, since G/H is compact, has a well defined trace. Consequently, the map

$$\Theta_T : \mathrm{C}^\infty_c(G) \ni f \mapsto \operatorname{Tr} T(f) \in \mathbb{C}$$

defines a distribution on G called the *distribution character of the induced representation* T. On the other hand, assume that $g \in G$ is such that $l_{g^{-1}} : G/H \to G/H$,

$xH \mapsto g^{-1}xH$, has only simple fixed points. In this case, a flat trace $\mathrm{Tr}^\flat T(g)$ of $T(g)$ can be defined according to

$$\mathrm{Tr}^\flat T(g) = \mathrm{Tr}_{l_{g^{-1}}}\big(\Gamma(\varphi_g)\big),$$

where $\varphi_g : l^*_{g^{-1}}(G \times_H V) \to G \times_H V$ is the endomorphism associated to $T(g)$ such that $T(g) = \varphi_g \circ l^*_{g^{-1}}$, and $\Gamma(\varphi_g) : \Gamma(l^*_{g^{-1}}(G \times_H V)) \to \Gamma(G \times_H V)$. $\mathrm{Tr}^\flat T(g)$ is given by a sum over fixed points of g, and one can show that, on an open set $G_T \subset G$,

$$\Theta_T(f) = \int\limits_{G_T} f(g)\,\mathrm{Tr}^\flat T(g) d_G(g), \quad f \in \mathrm{C}^\infty_\mathrm{c}(G_T).$$

Thus, the distribution character of a parabolically induced representation of a Lie group G is represented on G_T by the flat trace of the corresponding geometric endomorphism. If G is compact, the Lefschetz theorem reduces to the Hermann–Weyl formula by the theory of Borel and Weil. It can be interpreted as expressing the character of a finite dimensional representation as an alternating sum of characters of infinite dimensional representations.

In what follows, we shall prove similar formulae for the distributions Θ_π and Θ^s_π defined in Section 6. Let the notation be as before, and denote by $\Phi_g(\tilde{x}) = g^{-1} \cdot \tilde{x}$ the G-action on $\widetilde{\mathbb{X}}$. Note that the set $G(\widetilde{\mathbb{X}}) = \{g \in G : \Phi_g \text{ is transversal}\} \subset G$ of elements acting transversally on $\widetilde{\mathbb{X}}$ is open. Furthermore, Corollary 3 and Remark 6 imply that $G(\widetilde{\mathbb{X}})$ is dense if $\mathrm{rank}(G/K) = 1$.

Theorem 7. *Let $f \in \mathrm{C}^\infty_\mathrm{c}(G)$ have support in $G(\widetilde{\mathbb{X}})$, and $s \in \mathbb{C}$, $\mathrm{Re}\, s > -1$. Then*

$$\mathrm{Tr}_s\, \pi(f) = \int\limits_{G(\widetilde{\mathbb{X}})} f(g)\Bigg(\sum_{\tilde{x} \in \mathrm{Fix}(\widetilde{\mathbb{X}},g)} \sum_\gamma \frac{\alpha_\gamma(\tilde{x})|x_{k+1}(\kappa_\gamma^{-1}(\tilde{x}))\cdots x_{k+l}(\kappa_\gamma^{-1}(\tilde{x}))|^{s+1}}{|\det(\mathbf{1} - d\Phi_g(\tilde{x}))|}\Bigg) d_G(g), \tag{42}$$

where $\mathrm{Fix}(\widetilde{\mathbb{X}}, g)$ *denotes the set of fixed points of Φ_g on $\widetilde{\mathbb{X}}$. In particular, $\Theta^s_\pi : \mathrm{C}^\infty_\mathrm{c}(G) \ni f \to \mathrm{Tr}_s\, \pi(f) \in \mathbb{C}$ is regular on $G(\widetilde{\mathbb{X}})$.*

Proof. By Proposition 2,

$$\mathrm{Tr}_s\, \pi(f) = \sum_\gamma \int\limits_{W_\gamma} (\alpha_\gamma \circ \varphi_\gamma)(x)|x_{k+1}\cdots x_{k+l}|^s \widetilde{A}^\gamma_f(x,0) dx$$

is a meromorphic function in s with possible poles at $-1, -3, \ldots$. Assume that $\mathrm{Re}\, s > -1$. Since $\alpha_\gamma \in \mathrm{C}^\infty_\mathrm{c}(\widetilde{W}_\gamma)$, and $\widetilde{A}^\gamma_f(x,0) = \int \tilde{a}^\gamma_f(x,\xi) đ\xi$, where $\tilde{a}^\gamma_f(x,\xi) \in \mathrm{S}^{-\infty}_{la}(W_\gamma \times \mathbb{R}^{k+l})$ is rapidly decaying in ξ by Theorem 2, we can interchange the order of integration to obtain

$$\mathrm{Tr}_s\,\pi(f)=\sum_\gamma\int\int_{W_\gamma}(\alpha_\gamma\circ\varphi_\gamma)(x)|x_{k+1}\cdots x_{k+l}|^s\tilde{a}_f^\gamma(x,\xi)\,dx\,đ\xi.$$

Let $\chi\in\mathrm{C}_{\mathrm{c}}^\infty(\mathbb{R}^{k+l},\mathbb{R}^+)$ be equal 1 in a neighborhood of 0, and $\varepsilon>0$. Then, by Lebesgue's theorem on bounded convergence,

$$\mathrm{Tr}_s\,\pi(f)=\lim_{\varepsilon\to 0}I_\varepsilon,$$

where we defined

$$I_\varepsilon=\sum_\gamma\int\int_{W_\gamma}(\alpha_\gamma\circ\varphi_\gamma)(x)|x_{k+1}\cdots x_{k+l}|^s\tilde{a}_f^\gamma(x,\xi)\chi(\varepsilon\xi)\,dx\,đ\xi.$$

Taking into account (14), and interchanging the order of integration once more, one sees that

$$I_\varepsilon=\int_G f(g)\sum_\gamma\int\int_{W_\gamma}e^{i\Psi_\gamma(g,x)\cdot\xi}c_\gamma(x,g)(\alpha_\gamma\circ\varphi_\gamma)(x)|x_{k+1}\cdots x_{k+l}|^s\chi(\varepsilon\xi)\,dx\,đ\xi\,d_G(g),$$

everything in sight being absolutely convergent. Let us now set

$$I_\varepsilon(g)=f(g)\sum_\gamma\int\int_{W_\gamma}e^{i\Psi_\gamma(g,x)\cdot\xi}c_\gamma(x,g)(\alpha_\gamma\circ\varphi_\gamma)(x)|x_{k+1}\cdots x_{k+l}|^s\chi(\varepsilon\xi)\,dx\,đ\xi,$$

so that $I_\varepsilon=\int_G I_\varepsilon(g)d_G(g)$. We would like to pass to the limit under the integral, for which we are going to show that $\lim_{\varepsilon\to 0}I_\varepsilon(g)$ is an integrable function on G. For this, let us fix an arbitrary $g\in G(\widetilde{\mathbb{X}})$. By definition, Φ_g acts only with simple fixed points on $\widetilde{\mathbb{X}}$. Since each of them is isolated, Φ_g can have at most finitely many fixed points on $\widetilde{\mathbb{X}}$. Consider therefore a cut-off function $\beta_g\in\mathrm{C}^\infty(\widetilde{\mathbb{X}},\mathbb{R}^+)$ which is equal 1 in a small neighborhood of each fixed point of Φ_g, and whose support decomposes into a disjoint union of connected components, each of which contains only one fixed point of Φ_g. By choosing the support of β_g sufficiently close to the fixed points we can, in addition, assume that

$$\det\bigl(\mathbf{1}-d\Phi_g(\tilde{x})\bigr)\neq 0\quad\text{on }\operatorname{supp}\beta_g. \tag{43}$$

Since the action of G is real analytic, we obtain a family of functions $\beta_g(\tilde{x})$ depending smoothly on $g\in G(\widetilde{\mathbb{X}})$. Multiplying the integrand of $I_\varepsilon(g)$ with $\beta_g\circ\varphi_\gamma(x)$, and $1-\beta_g\circ\varphi_\gamma(x)$, respectively, we obtain the decomposition

$$I_\varepsilon(g)=I_\varepsilon^{(1)}(g)+I_\varepsilon^{(2)}(g).$$

Let us first examine what happens away from the fixed points. Integrating by parts $2N$ times with respect to ξ yields

$$\begin{aligned} I_\varepsilon^{(2)}(g) &= f(g) \sum_\gamma \int \int_{W_\gamma} e^{i\Psi_\gamma(g,x)\cdot\xi} c_\gamma(x,g)\big(\alpha_\gamma(1-\beta_g)\big)\big(\varphi_\gamma(x)\big)|x_{k+1}\cdots x_{k+l}|^s \chi(\varepsilon\xi)\,dx\,đ\xi \\ &= f(g) \sum_\gamma \int \int_{W_\gamma} \frac{e^{i\Psi_\gamma(g,x)\cdot\xi}}{|\Psi_\gamma(g,x)|^{2N}} \Delta_\xi^N \big[\chi(\varepsilon\xi)\big] \\ &\quad \times c_\gamma(x,g)\big(\alpha_\gamma(1-\beta_g)\big)\big(\varphi_\gamma(x)\big)|x_{k+1}\cdots x_{k+l}|^s \,dx\,đ\xi, \end{aligned}$$

where $\Delta_\xi = \partial_{\xi_1}^2 + \cdots + \partial_{\xi_{k+l}}^2$. Now, for arbitrary N,

$$\big|\Delta_\xi^N\big[\chi(\varepsilon\xi)\big]\big| \leqslant C_N\big(1+|\xi|^2\big)^{-N},$$

where C_N does not depend on ε for $0<\varepsilon\leqslant 1$, but certainly on the order of differentiation. Furthermore, there exists a constant $M_f>0$ such that $|\Psi_\gamma(g,x)|^{2N} \geqslant M_f$ on the support of $1-\beta_g\circ\varphi_\gamma$ for all $g\in\operatorname{supp} f$ and γ. By Lebesgue's theorem, we may therefore pass to the limit under the integral, and obtain

$$\lim_{\varepsilon\to 0} I_\varepsilon^{(2)}(g) = 0.$$

Hence, as $\varepsilon\to 0$, the main contributions to $I_\varepsilon(g)$ originate from the fixed points of Φ_g. To examine these contributions, note that condition (43) implies that $x\mapsto \varphi_\gamma^g(x)-x$ defines a diffeomorphism on each of the connected components of $\operatorname{supp}(\alpha_\gamma\beta_g)\circ\varphi_\gamma$ onto their respective images. Performing the change of variables $y = x-\varphi_\gamma^g(x)$ we get for $I_\varepsilon^{(1)}(g)$ the expression

$$\begin{aligned} & f(g)\sum_\gamma \int\int_{W_\gamma} e^{i\Psi_\gamma(g,x)\cdot\xi} c_\gamma(x,g)(\alpha_\gamma\beta_g)\big(\varphi_\gamma(x)\big)|x_{k+1}\cdots x_{k+l}|^s\chi(\varepsilon\xi)\,dx\,đ\xi \\ &\quad = f(g)\sum_\gamma \iint e^{-i(\mathbf{1}_k\otimes T_{x(y)}^{-1})y\cdot\xi}\big|x_{k+1}(y)\cdots x_{k+l}(y)\big|^s \\ &\qquad \times \frac{(\alpha_\gamma\beta_g)(\varphi_\gamma(x(y)))c_\gamma(x(y),g)}{|\det(\mathbf{1}-d\varphi_\gamma^g(x(y)))|}\chi(\varepsilon\xi)\,dy\,đ\xi \\ &\quad = f(g)\sum_\gamma \int \big|x_{k+1}(y)\cdots x_{k+l}(y)\big|^s c_\gamma\big(x(y),g\big)\frac{(\alpha_\gamma\beta_g)(\varphi_\gamma(x(y)))\hat\chi((\mathbf{1}_k\otimes T_{x(y)}^{-1})y/\varepsilon)}{(2\pi)^{k+l}\varepsilon^{k+l}|\det(\mathbf{1}-d\varphi_\gamma^g(x(y)))|}dy \\ &\quad = f(g)\sum_\gamma \int \big|x_{k+1}(\varepsilon y)\cdots x_{k+l}(\varepsilon y)\big|^s c_\gamma\big(x(\varepsilon y),g\big) \\ &\qquad \times \frac{(\alpha_\gamma\beta_g)(\varphi_\gamma(x(\varepsilon y)))\hat\chi((\mathbf{1}_k\otimes T_{x(\varepsilon y)}^{-1})y)}{(2\pi)^{k+l}|\det(\mathbf{1}-d\varphi_\gamma^g(x(\varepsilon y)))|}dy. \end{aligned}$$

Since in a neighborhood of a fixed point $\tilde{x}$ of g the Jacobian of the singular change of coordinates $z = (\mathbf{1}_k \otimes T^{-1}_{x(\varepsilon y)})y$ converges to the expression $|x_{k+1}(\kappa_\gamma^{-1}(\tilde{x})) \cdots x_{k+l}(\kappa_\gamma^{-1}(\tilde{x}))|^{-1}$ as $\varepsilon \to 0$, we finally obtain with $(2\pi)^{-k-l} \int \hat{\chi}(y)dy = \chi(0) = 1$ that

$$\begin{aligned}
\lim_{\varepsilon\to 0} I_\varepsilon^{(1)}(g) &= \lim_{\varepsilon \to 0} f(g) \sum_\gamma \int \bigl|x_{k+1}\bigl(\varepsilon y(z)\bigr) \cdots x_{k+l}\bigl(\varepsilon y(z)\bigr)\bigr|^s c_\gamma\bigl(x\bigl(\varepsilon y(z)\bigr), g\bigr) \\
&\quad \times \frac{(\alpha_\gamma \beta_g)(\varphi_\gamma(x(\varepsilon y(z))))|\partial y/\partial z|}{(2\pi)^{k+l}|\det(\mathbf{1} - d\varphi_\gamma^g(x(\varepsilon y(z))))|} \hat{\chi}(z)dz \\
&= f(g) \sum_{\tilde{x} \in \mathrm{Fix}(\widetilde{\mathbb{X}}, g)} \sum_\gamma \frac{\alpha_\gamma(\tilde{x})|x_{k+1}(\kappa_\gamma^{-1}(\tilde{x})) \cdots x_{k+l}(\kappa_\gamma^{-1}(\tilde{x}))|^{s+1}}{|\det(\mathbf{1} - d\Phi_g(\tilde{x}))|},
\end{aligned}$$

since $\bar{\alpha}_\gamma \equiv 1$ on $\operatorname{supp} \alpha_\gamma$, and $\beta_g(\tilde{x}) = 1$. The limit function $\lim_{\varepsilon\to 0} I_\varepsilon(g)$ is therefore clearly integrable on G for $\operatorname{Re} s > -1$, so that by passing to the limit under the integral one computes

$$\begin{aligned}
\operatorname{Tr}_s \pi(f) &= \lim_{\varepsilon\to 0} I_\varepsilon = \lim_{\varepsilon \to 0} \int_G I_\varepsilon(g) d_G(g) = \int_G \lim_{\varepsilon\to 0} \bigl(I_\varepsilon^{(1)} + I_\varepsilon^{(2)}\bigr)(g) d_G(g) \\
&= \int_G f(g) \sum_{\tilde{x} \in \mathrm{Fix}(\widetilde{\mathbb{X}}, g)} \sum_\gamma \frac{\alpha_\gamma(\tilde{x})|x_{k+1}(\kappa_\gamma^{-1}(\tilde{x})) \cdots x_{k+l}(\kappa_\gamma^{-1}(\tilde{x}))|^{s+1}}{|\det(1 - d\Phi_g(\tilde{x}))|} d_G(g),
\end{aligned}$$

yielding the desired description of Θ_π^s. $\square$

As an immediate consequence of the previous theorem, we see that if $f \in \mathrm{C}_\mathrm{c}^\infty(G(\widetilde{\mathbb{X}}))$, $\operatorname{Tr}_s \pi(f)$ is not singular at $s = -1$. This observation leads to the following

Corollary 4. *Let $f \in \mathrm{C}_\mathrm{c}^\infty(G)$ have support in $G(\widetilde{\mathbb{X}})$. Then*

$$\operatorname{Tr}_{reg} \pi(f) = \operatorname{Tr}_{-1} \pi(f) = \int_{G(\widetilde{\mathbb{X}})} f(g) \sum_{\tilde{x} \in \mathrm{Fix}(\widetilde{\mathbb{X}}, g)} \frac{1}{|\det(1 - d\Phi_g(\tilde{x}))|} d_G(g).$$

In particular, the distribution $\Theta_\pi : f \to \operatorname{Tr}_{reg}(f)$ is regular on $G(\widetilde{\mathbb{X}})$.

Proof. Consider the Laurent expansion of $\Theta_\pi^s(f)$ at $s = -1$ given by

$$\begin{aligned}
\operatorname{Tr}_s \pi(f) &= \Bigl\langle |x_{k+1} \cdots x_{k+l}|^s, \sum_\gamma (\alpha_\gamma \circ \varphi_\gamma) \widetilde{A}_f^\gamma(\cdot, 0) \Bigr\rangle \\
&= \sum_{j=-q}^\infty S_j \Bigl(\sum_\gamma (\alpha_\gamma \circ \varphi_\gamma) \widetilde{A}_f^\gamma(\cdot, 0)\Bigr)(s+1)^j,
\end{aligned}$$

where $S_k \in \mathcal{D}'(\mathbb{R}^{k+l})$. Since by (42), $\operatorname{Tr}_s \pi(f)$ has no pole at $s = -1$, we necessarily must have

$$S_j\Big(\sum_\gamma (\alpha_\gamma \circ \varphi_\gamma)\widetilde{A}_f^\gamma(\cdot, 0)\Big) = 0 \quad \text{for } j < 0,$$

so that

$$\operatorname{Tr}_{-1} \pi(f) = \Big\langle S_0, \sum_\gamma (\alpha_\gamma \circ \varphi_\gamma)\widetilde{A}_f^\gamma(\cdot, 0)\Big\rangle = \operatorname{Tr}_{reg} \pi(f).$$

The assertion now follows with the previous theorem. □

In particular, Corollary 4 implies that $\operatorname{Tr}_{reg} \pi(f)$ is invariantly defined. Now, interpreting $\pi(g)$ as a geometric endomorphism on the trivial bundle $E = \widetilde{\mathbb{X}} \times \mathbb{C}$ over the Oshima compactification $\widetilde{\mathbb{X}}$, a flat trace $\operatorname{Tr}^\flat \pi(g)$ of $\pi(g)$ can be defined according to

$$\operatorname{Tr}^\flat \pi(g) = \operatorname{Tr}_{\Phi_g}\big(\Gamma(\varphi_g)\big),$$

where $\varphi_g : \Phi_g^* E \to E$ is the associated bundle homomorphism which identifies the fiber $E_{\Phi_g(\tilde{x})}$ with $E_{\tilde{x}}$, and satisfies $(\operatorname{Tr} \varphi_g)_{|\tilde{x}} = 1$ at each fixed point $\tilde{x}$ of Φ_g. Taking into account (41), the previous corollary can be reformulated, and we finally deduce the following fixed point formula for the distribution character of π. In a future work, the authors hope to obtain a better understanding of this formula, and the contribution of the various orbit types to it.

Theorem 8. *On the set of transversal elements $G(\widetilde{\mathbb{X}})$, the distribution $\Theta_\pi : f \to \operatorname{Tr}_{reg} \pi(f)$ is given by*

$$\operatorname{Tr}_{reg} \pi(f) = \int_{G(\widetilde{\mathbb{X}})} f(g) \operatorname{Tr}^\flat \pi(g) d_G(g), \quad f \in \mathrm{C}_c^\infty\big(G(\widetilde{\mathbb{X}})\big),$$

where

$$\operatorname{Tr}^\flat \pi(g) = \sum_{\tilde{x} \in \operatorname{Fix}(\widetilde{\mathbb{X}}, g)} \frac{1}{|\det(1 - d\Phi_g(\tilde{x}))|},$$

the sum being over the (simple) fixed points of $g \in G(\widetilde{\mathbb{X}})$ on $\widetilde{\mathbb{X}}$.

We would like to finish by pointing out that, in obtaining our results, it was of crucial importance that the orbital decomposition of $\widetilde{\mathbb{X}}$ is of normal crossing type. Since wonderful varieties, such as the De-Concini Procesi compactification of a complexifed symmetric space, have the same kind of orbital decomposition, we expect that one could carry out a similar analysis on the real locus of such varieties, and introduce analogous distribution characters. In fact, it seems to us that such varieties are amenable towards a more refined understanding of these characters and the respective fixed point formulae in terms of combinatorial invariants.

References

[1] J.-P. Anker, L. Ji, Heat kernel and Green function estimates on noncompact symmetric spaces, Geom. Funct. Anal. 9 (6) (1999) 1035–1091.
[2] M.F. Atiyah, Resolution of singularities and division of distributions, Comm. Pure Appl. Math. 23 (1970) 145–150.
[3] M.F. Atiyah, R. Bott, A Lefschetz fixed point formula for elliptic complexes: I, Ann. of Math. 86 (1967) 374–407.
[4] M.F. Atiyah, R. Bott, A Lefschetz fixed point formula for elliptic complexes: II. Applications, Ann. of Math. 88 (1968) 451–491.
[5] I.N. Bernstein, S.I. Gel'fand, The meromorphic behavior of the function P^λ, Funct. Anal. Appl. 3 (1) (1969) 84–85.
[6] A. Borel, L. Ji, Compactifications of Symmetric and Locally Symmetric Spaces, Birkhäuser Boston Inc., Boston, 2006.
[7] W. Casselman, Canonical extensions of Harish-Chandra modules to representations of G, Can. J. Math. 41 (3) (1989) 385–438.
[8] L. Clozel, Théoréme d'Atiyah–Bott pour les variétés p-adiques et caractéres des groupes réductifs, Mem. Soc. Math. Fr. (N.S.) 15 (1984) 39–64.
[9] Harish-Chandra, Representations of semisimple Lie groups. III, Trans. Amer. Math. Soc. 76 (1954) 234–253.
[10] L. Hörmander, The Analysis of Linear Partial Differential Operators, vol. III, Springer-Verlag, Berlin, Heidelberg, New York, 1985.
[11] P. Loya, On the b-pseudodifferential calculus on manifolds with corners, PhD thesis, 1998.
[12] R.R. Mazzeo, R.B. Melrose, Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature, J. Funct. Anal. 75 (1987) 260–310.
[13] R.R. Mazzeo, A. Vasy, Analytic continuation of the resolvent of the Laplacian on $SL(3)/SO(3)$, Amer. J. Math. 126 (2004) 821–844.
[14] R.R. Mazzeo, A. Vasy, Analytic continuation of the resolvent of the Laplacian on symmetric spaces of noncompact type, J. Funct. Anal. 228 (2005) 311–368.
[15] R. Melrose, Transformation of boundary problems, Acta Math. 147 (1981) 149–236.
[16] T. Oshima, A realization of Riemannian symmetric spaces, J. Math. Soc. Japan 30 (1) (1978) 117–132.
[17] Aprameyan Parthasarathy, Analysis on the Oshima compactification of a Riemannian symmetric space of non-compact type, PhD thesis, Philipps Universität Marburg, 2012.
[18] P. Ramacher, Pseudodifferential operators on prehomogeneous vector spaces, Comm. Partial Differential Equations 31 (2006) 515–546.
[19] D.W. Robinson, Elliptic Operators and Lie Groups, Oxford University Press, Oxford, 1991.
[20] M.A. Shubin, Pseudodifferential Operators and Spectral Theory, second ed., Springer-Verlag, Berlin, Heidelberg, New York, 2001.
[21] N.R. Wallach, Asymptotic expansions of generalized matrix entries of representations of real reductive groups, in: Lie Group Representations I, Proc. Spec. Year, Univ. Md., College Park 1982–1983, in: Lect. Notes Math., vol. 1024, 1983, pp. 287–369.
[22] G. Warner, Harmonic Analysis on Semi-Simple Lie Groups, vol. I, Springer-Verlag, Berlin, Heidelberg, New York, 1972.