

D-branes, Exceptional Sheaves and Quivers on Calabi-Yau manifolds: From Mukai to McKay

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Abstract

We present a method based on mutations of helices which leads to the construction (in the large volume limit) of exceptional coherent sheaves associated with the ($\sum_a l_a = 0$) orbits in Gepner models. This is explicitly verified for a few examples including some cases where the ambient weighted projective space has singularities not inherited by the Calabi-Yau hypersurface. The method is based on two conjectures which lead to the analog, in the general case, of the Beilinson quiver for \mathbb{P}^n . We discuss how one recovers the McKay quiver using the gauged linear sigma model (GLSM) near the orbifold or Gepner point in Kähler moduli space.

1 Introduction

There has been significant progress in the study of D-branes on Calabi-Yau manifolds in the recent past following the construction of D-branes at Gepner points in Calabi-Yau moduli space. Among the significant results have been the description of the A- and B-type boundary states, corresponding to D-branes, in Gepner models and their transformation, under monodromy action as one moves in Kähler moduli space, to particular large-volume D-brane configurations [1–6]. But once explicit results from the Gepner points are available, one of the important issues is to understand how these descriptions of D-branes at Gepner points can be recovered in a reasonably straightforward and algorithmic manner from the description of D-branes on large-volume CY manifolds. More generally, it would be useful to derive an understanding of the D-brane spectrum and other related properties including aspects of their world-volume theories in all possible “phases” of CY manifolds. While the ability to perform explicit computations is the reason to study some points in the Kähler moduli space, like the Gepner points, the large-volume “phase” of CY manifolds provides a clearer geometric picture which is useful to understand in detail. As is well known in the closed string case, all “phases” can be reached by choosing appropriate values of the Fayet-Iliopoulos (FI) terms in the in the gauged linear sigma model (GLSM) [7]. Thus the GLSM seems to be the logical starting point for the study of D-branes in the different phases of the CY manifolds. Of course, in the case of D-branes, we will have to understand the problem of marginal stability, whereby D-branes present at one point in Kähler moduli space may decay into other branes as the moduli are varied. But nevertheless one may hope to address even these issues by some means through the use of the GLSM. In this paper it is this point of view, of trying to connect the behaviour of the D-brane spectrum in all the different phases by using the GLSM description of D-branes, that we will pursue.

Various aspects of such a connection between the D-brane spectrum in different phases, and in particular at the the large volume limit and the Gepner point have already begun emerging from the work of [3, 8–12]. Among the most important result of these studies has been the uncovering of the special role of classes of exceptional bundles (or more generally exceptional sheaves) on CY manifolds. These D-brane configurations (which have no moduli) appear to be the building blocks of other brane configurations with moduli. The D-brane configurations which are seen at the Gepner point by the Recknagel-Schomerus [1] construction are all in this class. The $\sum_a l_a = 0$ orbit of B-type boundary states are all related to exceptional bundles in the large volume limit by monodromy transformations. The other B-type boundary states seen at the Gepner point with $\sum_a l_a \neq 0$ are clearly bound states of these exceptional objects [10, 11].

The study of the case of the $\mathbb{C}^3/\mathbb{Z}_3$ and its blowup to \mathbb{P}^2 first showed a natural connection between the fractionally charged branes (or equivalently wrapped branes pinned to the orbifold point in \mathbb{C}^3) and a particular class of exceptional

bundles on \mathbb{P}^2 [3, 10]. Whereas the branes that exist at the orbifold point can be described by a quiver gauge theory, based on the McKay quiver¹, the corresponding bundles on the blowup are related to a truncated version of this quiver, namely the Beilinson quiver. Moreover the Beilinson quiver can be seen to be equivalent to the monad construction for bundles on \mathbb{P}^2 . Further, any sheaf on \mathbb{P}^2 can be written as a cohomology of a complex involving the bundles that are the large-volume monodromy transforms of the fractional branes in the orbifold limit. This permits the description of D-branes in the large volume limit in terms of the numbers naturally associated with the labelling of charges of D-branes in the orbifold limit. This relationship immediately provides a handle on the question of marginal stability. Whereas the charges of stable D-branes at the orbifold point (in the orbifold basis) have to be either all positive or all negative, the bundles in the large volume have no such restriction. Clearly the candidates for decay (in the passage) from the large-volume limit to the orbifold point, are those branes which have a combination of negative and positive charges in the orbifold basis.

The extension of this procedure to the study of the Landau-Ginzburg(LG) theory associated to the Gepner point of a CY manifold is a logical one. One may regard the LG theory as an orbifold of the type \mathbb{C}^n/Γ if one disregards the superpotential. Douglas and Diaconescu [11] (see also [3] for related observations) considered only the so-called tautological line bundles on the orbifold \mathbb{C}^n/Γ (this is equivalent to considering only the states associated with the vector representation of $Sl(n, \mathbb{C})$). Then by using the inverse procedure of reconstructing a toric variety from the quiver data associated with the orbifold (using the methods of [13, 14]), that one can construct the corresponding line bundles, $\{R_i\}$ on the space obtained by the resolution of the singularities of \mathbb{C}^n/Γ . One then constructs a set of vector bundles $\{S_i\}$ on the resolved space which are dual (in the sense of K-theory) to the $\{R_i\}$. The restriction of these $\{S_i\}$ to the CY hypersurface provides exactly the set of exceptional vector bundles $\{V_i\}$ associated with the $\sum_a l_a = 0$ orbit of D-branes at the Gepner point. More precisely, it was verified in [11] that the D-brane charges associated with the $\{V_i\}$ were exactly the same as those obtained by monodromy transformation of the D-brane charges of the $\sum_a l_a = 0$ orbit at the Gepner point.

However the methods used by Douglas and Diaconescu to derive this remarkable result have two obvious limitations. First, the inverse procedure of reconstructing the toric variety that corresponds to resolving the singularities of the orbifold at the Gepner point is a cumbersome one. More importantly, it is not clear that there is a canonically-defined procedure for all resolutions of the singularities. Generically the CY hypersurface may not intersect some of the singularities and it is not clear how one may deal with such situations.

It is the attempt of this paper to show that in the frame-work of the GLSM

¹for earlier related work on quiver gauge theories and D-branes on orbifolds which are directly related to our context, see [13, 14].

there is a method to construct a set of objects analogous to the fundamental objects of the Douglas-Diaconescu construction², namely the $\{R_i\}$, $\{S_i\}$, and the $\{V_i\}$. The $\{V_i\}$ that we construct are identical to the set constructed by Diaconescu and Douglas in the examples they consider. The objects $\{R_i\}$ and $\{S_i\}$ are similar to theirs in the sense that the $\{R_i\}$ are a set of line bundles and that they are dual to the $\{S_i\}$ in the sense of intersection theory. Further the restriction of $\{S_i\}$ to the CY hypersurface produces exactly the $\{V_i\}$. Furthermore we construct the $\{R_i\}$ in a canonical manner using the GLSM in the neighbourhood of the large volume limit and show that the $\{S_i\}$ are defined by a consistent mathematical procedure that has a nice physical interpretation. The construction of the $\{V_i\}$ is then a straightforward matter. In a number of examples with one and two Kähler moduli we verify our claims. From the $\{R_i\}$ that we have constructed we can in fact reconstruct the quiver that is the analogue of the Beilinson quiver in the general case. We take a further step and argue that the presence of the so-called p -field in the GLSM allows us to reconstruct the McKay quiver that is associated to the orbifold LG theory, (which indeed may or may not have a Gepner construction).

In the next subsection we will sketch a more detailed overview of our method and results before we flood the reader with more technical details.

1.1 From Mukai to McKay

The natural starting point for the construction of rigid D-branes on CY manifolds is the 6-brane. Clearly there are no moduli in this case. In more mathematical terminology this is associated to the line bundle of degree zero on the CY manifold M , denoted by \mathcal{O}_M . We can obtain this line bundle by restricting the line bundle \mathcal{O} on the ambient variety X to the CY. We obtain an infinite set of other rigid objects $R_i, i \in \mathbb{Z}$, through the action of large-volume monodromy. The effect of the shift $B \rightarrow B + 1$ is implemented by the shift $\theta \rightarrow \theta + 2\pi$ in the GLSM. This has the effect of tensoring the initial bundle by a line bundle [18, 19]. This is physically equivalent to turning on D4-brane charges on the 6-brane together with lower brane charges.

Confining ourselves to the case where X is \mathbb{P}^n for the moment, we can restrict our considerations to the set $R_i = \mathcal{O}(i - 1)$, where i runs over 1 to $(n + 1)$. This collection of line bundles $\mathcal{R} \equiv \{R_i\}$ will be our starting point. The collection \mathcal{R} is referred to as the foundation of a helix for reasons that will be made clear later. From the work of Rudakov and others [20] we have a well-defined mathematical procedure, known as mutations, that enable us to construct the $\{S_i\}$. This procedure has a nice physical interpretation as has been demonstrated by Hori, Iqbal

²While this construction draws inspiration in an important way from the mathematical work of Ito and Nakajima [15], the authors' use of this work goes considerably beyond its original form. The reviews by Miles Reid of the McKay correspondence [16, 17] are also useful in this context.

and Vafa [19] (see also the earlier paper of Zaslow [21]). Mutations correspond to brane creation in the mirror of the original variety X . Take a neighbouring pair (R_i, R_{i+1}) in the set \mathcal{R} and consider the corresponding middle dimensional cycles on the mirror. In the mirror, a mutation corresponds to the creation of a new middle dimensional brane when the R_i “crosses” R_{i+1} . The mirror of this new middle-dimensional brane in the original variety X is referred to as the mutation of the two R_i that were our starting point.

A sequence of such mutations enables us to construct all the $\{S_i\}$ up to a sign that is fixed so that the $\{S_i\}$ are in fact dual to the $\{R_i\}$. If the Chern classes of the $\{S_i\}$ are all that we are interested in then of course this can be obtained by simply examining the inverse of the Euler matrix of the $\{R_i\}$, as is indeed clear from the work of Douglas and Diaconescu. On restricting the $\{S_i\}$ to the CY hypersurface, one obtains the $\{V_i\}$.

For the general case, going beyond \mathbb{P}^n , we formulate two conjectures that we verify in a variety of one and two Kähler modulus examples.

Conjecture 1: The large volume monodromy action on \mathcal{O} in the ambient variety X produces an exceptional collection which is the foundation of a helix of appropriate period p :

$$\mathcal{R} = (R_1 = \mathcal{O}, R_2, \dots, R_p) \tag{1.1}$$

Conjecture 2: All exceptional bundles and sheaves on X can be obtained by the mutations of the helix \mathcal{R} given in Conjecture 1. In particular, there exists a mutated helix $\mathcal{S} = (S_p, \dots, S_1 = \mathcal{O})$ with $S_i = L^{i-1}(R_i)$, where L^i corresponds to a sequence of i left-mutations.

Conjecture 2a: All the exceptional bundles on the CY which correspond to the $\sum_a l_a = 0$ states at the Gepner point are obtained by the restriction of the S_i to the CY hypersurface.

Indeed since exceptional collections and helices are known for a wide range of varieties including Grassmannians, products of projective and flag varieties etc. (see [20] and references therein) it appears that we can carry out this procedure for a wide range of CY manifolds that can be described as hypersurfaces in such ambient varieties. (In the cases where there is no corresponding Gepner construction, we would have to suitably generalise conjecture 2a).

The corresponding quiver gauge theory can now be written down in a canonical way. It is interesting that we have circumvented the problem of determining which resolution of the singularities of the orbifold associated with the Gepner point to deal with, since the large-volume limit of the GLSM provides us with a definite resolution that we must deal with. However this quiver is the analogue of the Beilinson quiver and misses the symmetry at the Gepner point that is due to the orbifold group. Restoring this symmetry corresponds to going to the corresponding McKay quiver from the analogue of the Beilinson quiver. We argue that this is done by the p -field. It is the p -field that enables the missing link to

be restored in the quiver diagram that then becomes the McKay quiver. The link was missing in the constructions of the large-volume limit since the p -field is set to zero in that limit. We leave the detailed argument to a subsequent section of this paper. This rule provides us, in principle, a means to write down the quiver gauge theories corresponding to the world-volume theories of the branes in different phases of the CY manifold, depending on which of the several p -fields are set to zero or acquire a non-zero vacuum expectation value in the world-sheet theory. Finally, by using a generalisation of Beilinson’s methods for \mathbb{P}^n , one can in principle construct all sheaves on the ambient projective space. The restriction of these sheaves to the Calabi-Yau hypersurface M provides one with a large collection of sheaves on M .

1.2 Motivation

We would like to explain the motivation and background that have led to the results that are described in this paper. The work arose as a result of trying to understand D-branes (in particular, D-branes with B-type boundary conditions) in gauged linear sigma models with boundary. In particular, the theta term in the GLSM Lagrangian imposes non-trivial constraints on the possible boundary conditions. Further, one needs to add a boundary *contact* term in order that boundary conditions in the GLSM have a proper large volume (NLSM) limit [18]. For the case of a six-brane, i.e., the D-brane associated with line bundle \mathcal{O} , one can see that this contact term ensures that the large-volume monodromy is captured properly in the GLSM with boundary. This, in fact, carries over to the case of D-branes associated with non-trivial vector bundles. Thus, the importance of large-volume monodromy is best seen in the GLSM with boundary [22].

As is well known, the GLSM interpolates between different phases of the Calabi-Yau manifold. Even in the simple example of Calabi-Yau manifolds with one Kähler modulus, a key addition in the GLSM to the fields which exist in the NLSM limit is the so-called p -field whose vacuum expectation value (vev) plays the role of an order parameter. In considerations of the GLSM with boundary, this field again plays an important role. As we will see, even if one stays in the geometric phase, the transition from the NLSM to the GLSM provides extra information about other phases as well provided we account for the behaviour of the p -field.

Mirror symmetry (in its extended form) also plays an important role in our considerations though they are not quite required for the results. The large volume limit gets mapped to a particular point in the moduli space of complex structures on the mirror. This point is rather special – it is the point of *maximal unipotent monodromy* and is given by the transverse intersection of divisors corresponding to degeneration of Calabi-Yau manifolds (see [23] and references therein for more details). Further, the process of mutations has been related to brane creation on the mirror [19].

The paper is organised as follows: In section 2, we set up our notation as well as conventions. In section 3, we define helices and mutations and then describe our conjectures. As a result, we give a precise method to obtain the $\sum_a l_a = 0$ bundles for Gepner models associated with Calabi-Yau manifolds. In section 4, we test our conjectures in several examples with one and two Kähler moduli. We also show how one deals with singularities (in the ambient weighted projective space) that are not inherited by Calabi-Yau hypersurfaces. In section 5, we show how one constructs the generalised Beilinson quiver from the helices and then argue how the p -field enables one to recover the McKay quiver associated with the LG orbifold/Gepner model. We conclude in section 6 with some remarks.

2 Notation and Conventions

We will be consider two classes of models in this paper. We shall use the language of the gauged linear sigma model (GLSM)/toric description in the discussion of the models

2.1 The one-parameter models

The GLSM consists of six chiral superfields Φ_i ($i = 0, 1, \dots, 5$) and a single abelian vector superfield. The bosonic components ϕ_i of the chiral superfields will later be identified with the quasi-homogeneous coordinates of some weighted projective space. The Fayet-Iliopoulos parameter is labelled $t = r + i\frac{\theta}{2\pi}$ and is related to the complexified Kähler modulus ($B + iV$). The chiral superfield Φ_0 will also be called the P -field and plays a special role. The charge vector is

$$q_i = (q_0 \quad q_1 \quad q_2 \quad q_3 \quad q_4 \quad q_5) \quad ,$$

with $q_0 = -\sum_{i=1}^5 q_i$. The D-term constraint is given by

$$D = -e^2 \left(\sum_{i=0}^5 q_i |\phi_i|^2 - r \right) \quad ,$$

In the absence of a superpotential, in the large volume limit, the space is a non-compact Calabi-Yau manifold corresponding to the total space of the line bundle $\mathcal{O}(q_0)$ over the weighted projective space $\mathbb{P}^{q_1, q_2, q_3, q_4, q_5}$.

The introduction of a superpotential $W = PG(\Phi)$, where G is a quasi-homogeneous function (of Φ_i for $i \neq 0$) of degree $|q_0|$, gives rise to a compact Calabi-Yau manifold as a hypersurface M given by the equation $G = 0$ in weighted projective space $X \equiv \mathbb{P}^{q_1, q_2, q_3, q_4, q_5}$ in the NLSM limit ($e^2 \sqrt{r} \rightarrow \infty$)³. Note that, in the NLSM limit, the field p is set to zero in the ground state.

³We shall hereafter refer to the ambient weighted projective space as X and the compact Calabi-Yau hypersurface as M . We will also use the terms *large volume limit* and the *NLSM limit* in an interchangeable manner.

We shall consider four specific examples given by (studied in [24, 25])

$$\mathbb{P}^4[5] = \mathbb{P}^{1,1,1,1,1}[5] : \quad G = \phi_1^5 + \phi_2^5 + \phi_3^5 + \phi_4^5 + \phi_5^5 = 0 \quad , \quad (2.1)$$

$$\mathbb{P}^{1,1,1,1,2}[6] : \quad G = \phi_1^6 + \phi_2^6 + \phi_3^6 + \phi_4^6 + \phi_5^3 = 0 \quad , \quad (2.2)$$

$$\mathbb{P}^{1,1,1,1,4}[8] : \quad G = \phi_1^8 + \phi_2^8 + \phi_3^8 + \phi_4^8 + \phi_5^2 = 0 \quad , \quad (2.3)$$

$$\mathbb{P}^{1,1,1,2,5}[10] : \quad G = \phi_1^{10} + \phi_2^{10} + \phi_3^{10} + \phi_4^5 + \phi_5^2 = 0 \quad . \quad (2.4)$$

Weighted projective spaces are typically singular. This is true of all the examples we consider except for the case of \mathbb{P}^4 . For example, consider $\mathbb{P}^{1,1,1,1,2}$. In the chart $\phi_5 = 1$, there is a \mathbb{Z}_2 identification:

$$(\phi_1, \phi_2, \phi_3, \phi_4) \sim (-\phi_1, -\phi_2, -\phi_3, -\phi_4)$$

and thus one has a \mathbb{Z}_2 orbifold singularity at the origin. One can however verify that the singularity is not a point in the compact Calabi-Yau manifold and hence the hypersurface does not inherit the singularity from the ambient projective space. In a similar fashion, $\mathbb{P}^{1,1,1,1,4}$ has an \mathbb{Z}_4 orbifold singularity in the origin of the chart $\phi_5 = 1$ and $\mathbb{P}^{1,1,1,2,5}$ has two orbifold singularities: a \mathbb{Z}_5 in the chart $\phi_5 = 1$ and a \mathbb{Z}_2 in the chart $\phi_4 = 1$. In all these cases, the hypersurface does not inherit the singularities and hence one obtains a smooth Calabi-Yau manifold.

There might be other situations where the hypersurface does inherit the singularity and one has the option of blowing up the singularity by, say a \mathbb{P}^1 . This naturally leads us to models with two Kähler moduli where the second modulus is associated with “size” of the blown up \mathbb{P}^1 .

2.2 Two parameter examples

The GLSM involves seven chiral superfields Φ_i and two abelian vector superfields. We shall again refer to Φ_0 as the P -field. There are two Kähler moduli which are related to the Fayet-Iliopoulos terms which we shall label $t_a = r_a + i\frac{\theta_a}{2\pi}$, for $a = 1, 2$. The D-term constraints (moment maps) are

$$D^a = -e^2 \left(\sum_{i=0}^5 q_i |\phi_i|^2 - r_a \right) \quad ,$$

We shall consider two models (these have been studied in [26]):

1. The charge vectors are

$$q_i^a = \begin{pmatrix} -4 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 \end{pmatrix} \quad ,$$

and the model is the blowup of the weighted projective space $\mathbb{P}^{1,1,2,2,2}$. The compact Calabi-Yau is a hypersurface of bidegree $(4, 0)$ such as

$$\mathbb{P}^{1,1,2,2,2}[8] : \quad G = \phi_1^8 \phi_6^4 + \phi_2^8 \phi_6^4 + \phi_3^4 + \phi_4^4 + \phi_5^4 = 0$$

2. The charge vectors are

$$q_i^a = \begin{pmatrix} -6 & 0 & 0 & 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 \end{pmatrix} ,$$

and the model is the blowup of the weighted projective space $\mathbb{P}^{1,1,2,2,6}$. However, it has a \mathbb{Z}_6 singularity which is not inherited by the hypersurface given by a transverse polynomial of bidegree $(6, 0)$ such as

$$\mathbb{P}^{1,1,2,2,6}[12] : \quad G = \phi_1^{12} \phi_6^6 + \phi_2^{12} \phi_6^6 + \phi_3^6 + \phi_4^6 + \phi_5^6 = 0$$

2.3 Geometric data

Let J_a ($a = 1, \dots, h_{1,1}$) be the Kähler classes of a Calabi-Yau manifold with the complexified Kähler cone given by $K = \sum_a t_a J_a$. The data associated with the special Kähler geometry associated with $(2, 2)$ Calabi-Yau compactifications is encoded in a prepotential $\mathcal{F}(t_a)$ which in the large-volume limit takes the general form [27]

$$\mathcal{F} = -\frac{1}{6} \kappa_{abc} t_a t_b t_c + \frac{1}{2} \alpha_{ab} t_a t_b + \beta_a t_a + \frac{1}{2} \gamma + \text{instanton corrections} \quad (2.5)$$

where α_{ab} , β_a and γ are constants which reflect an $\text{Sp}(2h_{1,1} + 2)$ ambiguity and κ_{abc} is the triple intersection

$$\kappa_{abc} = \int_M J_a \wedge J_b \wedge J_c \equiv \langle J_a J_b J_c \rangle_M \quad . \quad (2.6)$$

The associated period vector is given by

$$\vec{\Pi}(t) = \begin{pmatrix} \frac{1}{6} \kappa_{abc} t_a t_b t_c + \beta_a t_a + \gamma \\ -\frac{1}{2} \kappa_{abc} t_b t_c + \alpha_{ab} t_b + \beta_a \\ t_a \\ 1 \end{pmatrix} \quad (2.7)$$

where we have not included the instanton contributions which are suppressed in the large volume limit.

The central charge associated with a D-brane with RR charge vector $\vec{n} = (n_6, n_4^a, n_2^a, n_0)$ is given by⁴

$$Z(\vec{n}) = \vec{n} \cdot \vec{\Pi} \quad (2.8)$$

The central charge associated with a coherent sheaf E is given by [28]

$$Z(E) = \int_M e^{-t_a J_a} \wedge ch(E) \wedge \left(1 + \frac{c_2(M)}{24} \right) \quad (2.9)$$

⁴In the one parameter models, this choice differs in the choice of the sign of n_6 from those considered in [2, 6]. We account for this when we compare our results with theirs. We make this choice in order to have a uniform convention across all models.

By comparing the expressions (2.8) and (2.9) one obtains a map relating the Chern classes of E to the D-brane charges \vec{n} . We obtain

$$\begin{aligned}
\text{ch}_0(E) &= -n_6 \\
\text{ch}_1(E) &= -n_4^a J_a \\
\langle \text{ch}_2(E) J_b \rangle_M &= -n_2^b - n_4^a \alpha_{ab} + n_6 \left(\beta_b - \frac{\langle c_2(M) J_b \rangle_M}{24} \right) \\
\langle \text{ch}_3(E) \rangle_M &= n_0 + \left(\beta_b + \frac{\langle c_2(M) J_b \rangle_M}{24} \right) n_4^b + \gamma n_6
\end{aligned} \tag{2.10}$$

We will now impose the condition that the line bundle \mathcal{O} correspond to the pure anti-six brane i.e., $n_6 = -1$ and all other n 's being zero. This is satisfied by the choice

$$\begin{aligned}
\beta_b &= \frac{\langle c_2(M) J_b \rangle_M}{24} \\
\gamma &= 0
\end{aligned} \tag{2.11}$$

This does not fix the parameter α_{ab} . We will make model-dependent choices in order to make the two-brane charges n_2^a integers. This is similar to the requirement that the periods change by an integer symplectic matrix under the action $t_a \rightarrow t_a + 1$ for all a .

2.3.1 One parameter models

The intersection numbers for the one parameter models are [24, 25]:

$$\kappa = \langle J^3 \rangle_M = |q_0| \langle J^4 \rangle_X = \frac{|q_0|}{\prod_{i=1}^5 q_i} \tag{2.12}$$

Thus we have

$$\mathbb{P}^{1,1,1,1,1}[5] : \quad \kappa = 5, \quad \alpha = -\frac{11}{2}, \quad \beta = \frac{25}{12} \tag{2.13}$$

$$\mathbb{P}^{1,1,1,1,2}[6] : \quad \kappa = 3, \quad \alpha = -\frac{9}{2}, \quad \beta = \frac{7}{4} \tag{2.14}$$

$$\mathbb{P}^{1,1,1,1,4}[8] : \quad \kappa = 2, \quad \alpha = -3, \quad \beta = \frac{11}{6} \tag{2.15}$$

$$\mathbb{P}^{1,1,1,2,5}[10] : \quad \kappa = 1, \quad \alpha = -\frac{1}{2}, \quad \beta = \frac{17}{12} \tag{2.16}$$

In the above, we have chosen values for α as in ref. [24] where the choice enabled the large volume periods of M to be related to the periods (in the large complex structure limit) on the mirror W by means of an integer symplectic matrix. In our case, this seems to ensure that the RR-charges are integers.

2.3.2 Two parameter examples

The Kähler classes are generated by $H \equiv J_1$ and $L \equiv J_2$. The non-vanishing intersection numbers are

$$\kappa_{111} \equiv \langle H^3 \rangle_M = 2|q_0^1| \langle H^4 \rangle_X \quad (2.17)$$

$$\kappa_{112} \equiv \langle H^2 L \rangle_M = 2|q_0^1| \langle H^3 L \rangle_X \quad (2.18)$$

One also has the relation $L^2 = 0$ for both models. The data for the two models are

$$\mathbb{P}^{1,1,2,2,2}[8] : \quad \kappa_{111} = 2\kappa_{112} = 8, \quad \beta_1 = \frac{7}{3}, \quad \beta_2 = 1 \quad (2.19)$$

$$\mathbb{P}^{1,1,2,2,6}[12] : \quad \kappa_{111} = 2\kappa_{112} = 4, \quad \beta_1 = \frac{13}{6}, \quad \beta_2 = 1 \quad (2.20)$$

We will choose to set $\alpha_{ab} = 0$ in both examples.

3 Helices and Mutations

A coherent sheaf E on a variety X (of dimension n) is called *exceptional* if

$$\begin{aligned} \text{Ext}^i(E, E) &= 0 \quad , \quad i \geq 1 \\ \text{Ext}^0(E, E) &= \mathbb{C} \quad , \end{aligned}$$

where $\text{Ext}^i(E, F)$ is the sheaf-theoretic generalisation of the cohomology groups $H^i(X, E^* \otimes F)$ for vector bundles E and F . An ordered collection of exceptional sheaves $\mathcal{E} = (E_1, \dots, E_k)$ is called a *strongly exceptional collection*⁵ if for all $a < b$, one has

$$\begin{aligned} \text{Ext}^i(E_b, E_a) &= 0 \quad , \quad i \geq 0 \\ \text{Ext}^i(E_a, E_b) &= 0 \quad , \quad i \neq i_0 \quad , \end{aligned}$$

for some i_0 (which is typically zero). The alternating sum of dimensions of the groups $\text{Ext}^i(E, F)$ defines a bilinear product (we call this the *Euler form*)

$$\chi(E, F) = \sum_{i=0}^n (-1)^i \dim (\text{Ext}^i(E, F)) = \int_X \text{ch}(E^* \otimes F) \text{Td}(X) \quad (3.1)$$

For the case of vector bundles E and F , this is the Witten index associated with the Dolbeault operator on the bundle $E^* \otimes F$ and $\text{Td}(X)$ is the Todd class of the tangent bundle of X .

⁵This is somewhat different from the definition of Rudakov and is the one used by Bondal [29]. An exceptional collection is one for which the $\text{Ext}^i(E_a, E_b) = 0$ for $a > b$.

For an exceptional collection, the matrix (“the Euler matrix”)

$$I_{ab} \equiv \chi(E_a, E_b) \quad ,$$

is an upper-triangular matrix with ones on the diagonal. For a strongly exceptional collection, the non-zero entries are given by $(-)^{i_0} \dim \text{Ext}^{i_0}(E_a, E_b)$.

New exceptional collections can be generated from old ones by a process called *mutation*. A *right mutation* of an exceptional pair (E_a, E_{a+1}) in an exceptional collection is defined by

$$R_{a+1}(E_a, E_{a+1}) = (E_{a+1}, R_{E_{a+1}}(E_a)) \quad (3.2)$$

and a *left mutation* by

$$L_a(E_a, E_{a+1}) = (L_{E_a}(E_{a+1}), E_a) \quad (3.3)$$

where we have introduced two new sheaves $R_{E_{a+1}}(E_a)$ and $L_{E_a}(E_{a+1})$ which are defined through exact sequences (see [20] for details). For example, when $\text{Ext}^0(E_a, E_{a+1}) \neq 0$ and the (evaluation) map $\text{Ext}^0(E_a, E_{a+1}) \otimes E_a \rightarrow E_{a+1}$ is surjective, then $L_{E_a}(E_{a+1})$ is defined by

$$0 \rightarrow L_{E_a}(E_{a+1}) \rightarrow \text{Ext}^0(E_a, E_{a+1}) \otimes E_a \rightarrow E_{a+1} \rightarrow 0 \quad (3.4)$$

and when the map $\text{Ext}^0(E_a, E_{a+1}) \otimes E_a \rightarrow E_{a+1}$ is injective, one uses

$$0 \rightarrow \text{Ext}^0(E_a, E_{a+1}) \otimes E_a \rightarrow E_{a+1} \rightarrow L_{E_a}(E_{a+1}) \rightarrow 0 \quad (3.5)$$

Similarly, $R_{E_{a+1}}(E_a)$ is defined by (in the surjective case)

$$0 \rightarrow E_a \rightarrow \text{Ext}^0(E_a, E_{a+1})^* \otimes E_{a+1} \rightarrow R_{E_{a+1}}(E_a) \rightarrow 0 \quad (3.6)$$

The Chern characters of the new sheaves are given by

$$\begin{aligned} \pm \text{ch}(L_{E_a}(E_{a+1})) &= \text{ch}(E_{a+1}) - \chi(E_a, E_{a+1}) \text{ch}(E_a) \\ \pm \text{ch}(R_{E_{a+1}}(E_a)) &= \text{ch}(E_a) - \chi(E_a, E_{a+1}) \text{ch}(E_{a+1}) \end{aligned} \quad (3.7)$$

with the plus sign used for the injective case and the minus sign for the surjective case. Further, the collection is assumed to be strongly exceptional (with $i_0 = 0$) and hence $\chi(E_a, E_{a+1}) = \dim \text{Ext}^0(E_a, E_{a+1})$.

The mutation of a strongly exceptional collection may not continue to be strongly exceptional. If the mutated collection is also strongly exceptional, the mutation is called *admissible*. The mutations that we consider in this paper (in order to generate S_i) are assumed to be admissible though we do not always verify this explicitly.

An exceptional collection $(E_i, i \in \mathbb{Z})$ is called a *helix of period p* if for all s the following condition is satisfied:

All pairs $(E_{s-1}, E_s), (E_{s-2}, L^1(E_s)), \dots, (E_{s-p+1}, L^{p-2}(E_s))$ admit left mutations and $L^{p-1}(E_s) = E_{s-p}$.

Thus a sequence of $(p-1)$ left mutations of a helix brings one back to an element of the helix modulo a shift of p . Each collection $(E_i, E_{i+1}, \dots, E_{i+p})$ is called a *foundation* of the helix $\{E_i\}$. Any helix is determined uniquely by any of its foundations. One can also define the helix using right mutations. We shall henceforth use the term helix for the foundation of a helix since we will have no need to distinguish them.

3.1 An example

An example of a helix on \mathbb{P}^n is furnished by

$$\mathcal{R} = (\mathcal{O}, \dots, \mathcal{O}(n))$$

The Euler matrix is given by

$$I_{ab} = \binom{n+b-a}{b-a}$$

By a sequence of left mutations of \mathcal{R}^* one obtains a *mutated* helix

$$\begin{aligned} \mathcal{S} &= (L^n(\mathcal{O}(n)), L^{n-1}\mathcal{O}(n-1), \dots, \mathcal{O}) \\ &= (\Omega^n(n), \Omega^{n-1}(n-1), \dots, \mathcal{O}) \end{aligned}$$

where

$$L^p(\mathcal{O}(p)) \equiv L_{\mathcal{O}}L_{\mathcal{O}(1)} \cdots L_{\mathcal{O}(p-1)}(\mathcal{O}(p)) \quad .$$

The bundles which make up the helix \mathcal{S} can be seen to be the ones which form the $\sum_a l_a = 0$ orbit in the $\mathbb{C}^{n+1}/\mathbb{Z}_{n+1}$ orbifold⁶. The left mutation can thus be identified with the generator g of the quantum \mathbb{Z}_{n+1} symmetry at the orbifold point.

The dual of the Euler sequence (and related sequences) given by eqn. (4.10) can be rewritten in a form which suggests generalisations to more non-trivial situations:

$$0 \rightarrow L^p(\mathcal{O}(p)) \rightarrow \mathcal{O}^{\oplus \binom{n+1}{p}} \rightarrow L^{p-1}(\mathcal{O}(p)) \rightarrow 0 \quad (3.8)$$

where $\binom{n+1}{p}$ is to be identified with the Euler form $\chi(\mathcal{O}, L^{p-1}(\mathcal{O}(p-1)) \otimes \mathcal{O}(1))$.

3.2 Two conjectures

In the following, we will assume that the Calabi-Yau threefold M is given by the transverse intersection of hypersurfaces in some weighted projective space X . We

⁶ This statement is somewhat imprecise. The $\sum_a l_a = 0$ bundles for \mathbb{P}^2 are $(\mathcal{O}, -\Omega^1(1), \mathcal{O}(-1))$ where the minus sign reflects the K-theory class. This can be corrected by introducing a $(-)^n$ factor with the left mutation.

also assume that M is smooth in the sense that all singularities that it inherits from X are resolved.

Conjecture 1 *The large-volume monodromy action on \mathcal{O} in the ambient variety X produces an exceptional collection which is the foundation of a helix with appropriate period p*

$$\mathcal{R} = (R_1 = \mathcal{O}, R_2, \dots, R_p) \quad . \quad (3.9)$$

As is well known, the large-volume monodromy is typically given by tensoring of line bundles which do not change either the (large-volume) stability as well dimension of moduli space of vector bundles. The simplest exceptional bundle is the brane which wraps the full space i.e, \mathcal{O} . Thus, the large-volume monodromy action will generate a set of exceptional line-bundles. The non-trivial part of the conjecture is that this set leads to a helix.

We are motivated to consider the structure of helices and their mutations by the work of Hori, Iqbal and Vafa [19] and earlier work of Zaslow [21]. In this work, they show that mutations of exceptional collections have a well-defined physical interpretation in terms of brane creation in the mirror to X . Further, on the mirror side, the large complex structure limit (which is identified with the large volume limit under the mirror map) is the point (in the moduli space of complex structures) with *maximal unipotent monodromy* [23]. The two-parameter examples studied in [26] illustrate this issue. The helix structure that we see reflect this structure. The period of the helix is related to the quantum symmetry which appears as a classical symmetry on the mirror.

Conjecture 2 *All exceptional bundles/sheaves can be obtained by mutations of the helix \mathcal{R} given by Conjecture 1. In particular, there exists a mutated helix $\mathcal{S} = (S_p, \dots, S_1 = \mathcal{O})$ with $S_i = L^{i-1}(R_i)$, where L^{i-1} corresponds to a sequence of $(i-1)$ left-mutations.*

Conjecture 2a: *All the exceptional bundles on the CY which correspond to the $\sum_a l_a = 0$ states at the Gepner point are obtained by the restriction $S_i|_M \equiv V_i$ to the CY hypersurface.*

The motivation for this conjecture comes from the simple observation, that V_i is indeed the $\sum_a l_a = 0$ orbit for the case of the quintic in \mathbb{P}^4 . Further, it is a suggestive structure which works in other examples as well (as we will see in the next section). Another important motivation comes from the work of Douglas and Diaconescu which suggests a dual relationship between the bundles R_i and the bundles S_i i.e., $\chi(R_i, S_j) = \delta_{ij}$. The process of mutations can be seen to change the upper-triangular matrix $\chi(\mathcal{R}, \mathcal{R})$ into a diagonal one by mutating the helix which is the second argument of $\chi(\mathcal{R}, \mathcal{R})$ (as shown in appendix A). One obvious caveat is that on restriction to the CY hypersurface, new moduli may appear. While one expects no new moduli to appear to the restriction of \mathcal{O} , mirror symmetry predicts the quantum \mathbb{Z}_p symmetry which relate the other entries in the helix S and hence we expect the result to go through for these cases.

Conjecture 2 (Stronger form) *The helix \mathcal{R} generates the bounded derived category of coherent sheaves on X , $\mathcal{D}^b(\text{Coh}(X))$.*

Conjecture 2 (in both forms) is true for \mathbb{P}^n and seems to follow from the results of Bondal (see Theorem 4.1 in [29]) for spaces with very ample anticanonical class. This implies that there exists a generalisation of Beilinson's theorem [30] for the case of weighted projective spaces along the lines followed by [31,32] for \mathbb{P}^n . The existence of such a generalisation is also implicit in the work of Douglas and Diaconescu [11].

4 Testing the conjectures

We first discuss the case of $\mathbb{P}^{1,1,1,1,2}$ [6] where we explicitly show how one deals with the orbifold singularity in the ambient weighted projective space. This also illustrates the methods that we employ in all other examples.

4.1 $\mathbb{P}^{1,1,1,1,2}$ [6]

The Todd class for $X = \mathbb{P}^{q_1, q_2, q_3, q_4, q_5}$ is given by

$$\text{Td}(X) = \prod_{i=1}^5 \left(\frac{q_i J}{1 - e^{-q_i J}} \right)$$

For the one parameter models, we expand the Chern character of a sheaf E as

$$\text{ch}(E) = Q_0 + Q_1 J + Q_2 \frac{J^2}{2} + Q_3 \frac{J^3}{6} \quad .$$

The translation of the Q_a to the RR charges is

$$\begin{aligned} n_6 &= -Q_0 \\ n_4 &= -Q_1 \\ n_2 &= -\frac{\kappa}{2} Q_2 + \alpha Q_1 \\ n_0 &= \frac{\kappa}{6} Q_3 + 2\beta Q_1 \end{aligned} \tag{4.1}$$

where κ , α and β are as defined earlier.

The large volume monodromy is obtained by the operations $\theta \rightarrow \theta + 2\pi$ in the GLSM. For the one parameter models, one obtains $\mathcal{O} \rightarrow \mathcal{O}(J)$. Thus, conjecture one predicts that the line bundles $\mathcal{O}(bJ)$ should form a helix. However, as pointed out earlier, the ambient weighted projective space has a \mathbb{Z}_2 singularity which is not inherited by the Calabi-Yau hypersurface. The *naive* computation of Euler form gives rise to fractions. The occurrence of these fractions is related

to orbifold singularity. We are interested in getting rid of the contribution from the singularity. We modify the Euler form as follows⁷:

$$\widehat{\chi}(\mathcal{O}, \mathcal{O}(bJ)) \equiv \chi(\mathcal{O}, \mathcal{O}(bJ)) + \frac{(-)^b}{32} \quad (4.2)$$

By explicitly using the *modified* Euler form, we obtain the following helix \mathcal{R} of period six given by

$$\mathcal{R} = [\mathcal{O}, \mathcal{O}(J), \mathcal{O}(2J), \mathcal{O}(3J), \mathcal{O}(4J), \mathcal{O}(5J)]$$

The period of length six reflects the quantum \mathbb{Z}_6 symmetry of the model. The Euler matrix is

$$I_{ab} = \widehat{\chi}(R_a, R_b) = \begin{pmatrix} 1 & 4 & 11 & 24 & 46 & 80 \\ 0 & 1 & 4 & 11 & 24 & 46 \\ 0 & 0 & 1 & 4 & 11 & 24 \\ 0 & 0 & 0 & 1 & 4 & 11 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.3)$$

One can also explicitly verify the numbers by counting the number of degree one monomial, degree two monomials and so on. For example, there are four degree one monomials $(\phi_1, \phi_2, \phi_3, \phi_4)$ and eleven degree two monomials (ten quadratic in ϕ_i $i = 1, \dots, 4$ and ϕ_5) and so on.

We need to now verify the second conjecture i.e., the $\sum_a l_a = 0$ bundles are given by a sequence of left mutations. The bundle $-S_2 = L_{\mathcal{O}}(\mathcal{O}(J))$ is defined by the exact sequence

$$0 \rightarrow L_{\mathcal{O}}(\mathcal{O}(J)) \rightarrow \mathcal{O}^{\oplus 4} \xrightarrow{f} \mathcal{O}(J) \rightarrow 0 \quad (4.4)$$

where $f^i = \phi_i$ for $i = 1, 2, 3, 4$. This sequence is similar to the Euler sequence associated with \mathbb{P}^3 (with homogeneous coordinates ϕ_1, \dots, ϕ_4) and is a bundle of rank three. The next bundle is given by the exact sequence

$$0 \rightarrow S_3 \rightarrow \mathcal{O}^{\oplus 5} \rightarrow L_{\mathcal{O}}(\mathcal{O}(J)) \otimes \mathcal{O}(J) \rightarrow 0 \quad (4.5)$$

We now come to the next one in the sequence. This is an interesting one. One can see that $\text{Hom}[\mathcal{O}, S_3 \otimes \mathcal{O}(J) = L_{\mathcal{O}(1)}L_{\mathcal{O}(2)}\mathcal{O}(3)]$ vanishes. Thus, the map is injective unlike the earlier ones and we obtain the sequence

$$0 \rightarrow S_3 \otimes \mathcal{O}(J) \rightarrow S_4 \rightarrow 0 \quad (4.6)$$

⁷We do not explicitly calculate this contribution since it will take us too far from our objective. It suffices to note that one observes a \mathbb{Z}_2 pattern in the unmodified numbers associated with the Euler form and this simple modification leads to an upper triangular matrix with ones on the diagonal. The true justification for this is that we get sensible results on restricting the bundles to the Calabi-Yau hypersurface.

This implies that $S_4 = S_3 \otimes \mathcal{O}(J)$. The other bundles are obtained in similar fashion. Without getting into sequences involved, one can easily obtain the Chern character of the bundles by using the inverse of the Euler matrix given above (following the argument in the appendix) The inverse matrix is given by

$$I_{ab}^{-1} = \begin{pmatrix} 1 & -4 & 5 & 0 & -5 & 4 \\ 0 & 1 & -4 & 5 & 0 & -5 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ 0 & 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.7)$$

The Chern character of the bundles are

$$\begin{aligned} \text{ch}(S_2) &= \text{ch}(\mathcal{O}(J)) - 4 \\ \text{ch}(S_3) &= \text{ch}(\mathcal{O}(2J)) - 4\text{ch}(\mathcal{O}(J)) + 5 \\ \text{ch}(S_4) &= \text{ch}(\mathcal{O}(3J)) - 4\text{ch}(\mathcal{O}(2J)) + 5\text{ch}(\mathcal{O}(J)) \\ \text{ch}(S_5) &= \text{ch}(\mathcal{O}(4J)) - 4\text{ch}(\mathcal{O}(3J)) + 5\text{ch}(\mathcal{O}(2J)) - 5 \\ \text{ch}(S_6) &= \text{ch}(\mathcal{O}(5J)) - 4\text{ch}(\mathcal{O}(4J)) + 5\text{ch}(\mathcal{O}(3J)) - 5\text{ch}(\mathcal{O}(2J)) + 4 \end{aligned}$$

Note that $S_6 = L^5(\mathcal{O}(5J)) = \mathcal{O}(-J)$ which shows that the helix has period six. Further, there is a version of Serre duality which takes the form $S_i \simeq S_{p+1-i}^* \otimes S_p$, where p is the period of the helix. Let $V_i = S_i|_M$ be the restriction of these bundles to the compact Calabi-Yau manifold M given by degree six hypersurface in the weighted projective space. A non-trivial test of our conjectures is to verify that the RR charges associated with these branes are indeed identical to the ones in the $\sum_a l_a = 0$ orbit. The Chern character of the $\sum_a l_a = 0$ bundles are predicted by our conjectures to be

$$\begin{aligned} \text{ch}(V_1) &= 1 \\ \text{ch}(V_2) &= -3 + J + \frac{J^2}{2} + \frac{J^3}{6} \\ \text{ch}(V_3) &= 2 - 2J + \frac{2J^3}{3} \\ \text{ch}(V_4) &= 2 - J^2 \\ \text{ch}(V_5) &= -3 + 2J - \frac{2J^3}{3} \\ \text{ch}(V_6) &= 1 - J + \frac{J^2}{2} - \frac{J^3}{6} \end{aligned} \quad (4.8)$$

The Chern character suggest the following relationships among the various bundles that appear: $V_4 = V_3^* \otimes \mathcal{O}(-J)$, $V_5 = V_2^* \otimes \mathcal{O}(-J)$ and $V_6 = \mathcal{O}(-J)$. The

RR charges are (in the convention $\vec{n} = (n_6, n_4, n_2, n_0)$)

$$\begin{aligned}
\vec{n}_1 &= (-1, 0, 0, 0) \\
\vec{n}_2 &= (3, -1, -6, 4) \\
\vec{n}_3 &= (-2, 2, 9, -5) \\
\vec{n}_4 &= (-2, 0, 3, 0) \\
\vec{n}_5 &= (3, -2, -9, 5) \\
\vec{n}_6 &= (-1, 1, 3, -4)
\end{aligned} \tag{4.9}$$

This is in agreement with the charges for the corresponding Gepner model [6] (modulo a sign difference in the six-brane charge). The non-trivial part of the check is that the six-charges that we obtain form a \mathbb{Z}_6 orbit given by the action of a matrix A (with $A^6 = 1$) with the property $\vec{n}_{i+1} = \vec{n}_i \cdot A$. This result clearly does not depend on the choice of α and reflects the quantum \mathbb{Z}_6 symmetry of the model. However, our choice of α ensures that the A is a matrix with integer entries.

4.2 Other one parameter models

4.2.1 The Quintic

The helix in \mathbb{P}^4 has period five (which reflects the quantum \mathbb{Z}_5 symmetry)

$$\mathcal{R} = [\mathcal{O}, \mathcal{O}(J), \mathcal{O}(2J), \mathcal{O}(3J), \mathcal{O}(4J)]$$

The mutated helix corresponding to the the $\sum_a l_a = 0$ orbit is given by

$$S_{i+1} = (-)^i L^i(\mathcal{O}(iJ)) = (-)^i \Omega^i(i)$$

where Ω is the cotangent bundle to \mathbb{P}^4 and Ω^p is the p -th exterior power of the cotangent bundle. One can verify that the definition of a left mutation leads to the following exact sequences which are derivable from the Euler sequence

$$0 \rightarrow \Omega^p(p) \rightarrow \mathcal{O}^{\oplus \binom{5}{p}} \rightarrow \Omega^{p-1}(p) \rightarrow 0 \tag{4.10}$$

The restriction of the above sequence to the quintic hypersurface gives the $\sum_a l_a = 0$ orbit as has already been observed in ref. [10].

4.2.2 $\mathbb{P}^{1,1,1,1,4}$ [8]

The foundation of a helix of period eight is found to be

$$\mathcal{R} = [\mathcal{O}, \mathcal{O}(J), \mathcal{O}(2J), \mathcal{O}(3J), \dots, \mathcal{O}(6J), \mathcal{O}(7J)]$$

In the above, we used the modified Euler form

$$\widehat{\chi}(\mathcal{O}, \mathcal{O}(bJ)) \equiv \chi(\mathcal{O}, \mathcal{O}(bJ)) + \text{orb}_4(b) \tag{4.11}$$

$$\text{orb}_4(b) = \begin{cases} -7/64 & \text{for } b = 0 \pmod{4} \\ -1/64 & \text{for } b = 1, 3 \pmod{4} \\ 9/64 & \text{for } b = 2 \pmod{4} \end{cases}$$

The term reflects the \mathbb{Z}_4 singularity in the weighted projective space $\mathbb{P}^{1,1,1,1,4}$. The Euler form is given by

$$I_{ab} = \begin{pmatrix} 1 & 4 & 10 & 20 & 36 & 60 & 94 & 140 \\ 0 & 1 & 4 & 10 & 20 & 36 & 60 & 94 \\ 0 & 0 & 1 & 4 & 10 & 20 & 36 & 60 \\ 0 & 0 & 0 & 1 & 4 & 10 & 20 & 36 \\ 0 & 0 & 0 & 0 & 1 & 4 & 10 & 20 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.12)$$

The Chern character of the $\sum_a l_a = 0$ bundles are

$$\begin{aligned} \text{ch}(V_1) &= 1 \\ \text{ch}(V_2) &= -3 + J + \frac{J^2}{2} + \frac{J^3}{6} \\ \text{ch}(V_3) &= 3 - 2J + \frac{2J^3}{3} \\ \text{ch}(V_4) &= -1 + J - \frac{J^2}{2} + \frac{J^3}{6} \end{aligned} \quad (4.13)$$

The Chern character of bundles have the pattern $\text{ch}V_{i+4} = -\text{ch}V_i$ and hence relate branes to anti-branes. We thus have given the Chern character for only the first four bundles. The Chern character suggest the following relationship among the bundles: $V_3 = -V_2^* \otimes \mathcal{O}(-J)$ and $V_4 = -\mathcal{O}(-J)$. The corresponding RR charges are

$$\begin{aligned} \vec{n}_1 &= (-1, 0, 0, 0) \\ \vec{n}_2 &= (3, -1, -4, 4) \\ \vec{n}_3 &= (-3, 2, 6, -6) \\ \vec{n}_4 &= (1, -1, -2, 4) \end{aligned} \quad (4.14)$$

Again, these agree with the corresponding Gepner model results modulo the sign of n_6 [6].

4.2.3 $\mathbb{P}^{1,1,1,2,5}$ [10]

The helix of period ten is found to be

$$\mathcal{R} = [\mathcal{O}, \mathcal{O}(J), \mathcal{O}(2J), \mathcal{O}(3J), \dots, \mathcal{O}(8J), \mathcal{O}(9J)]$$

after using the modified Euler form

$$\widehat{\chi}(\mathcal{O}, \mathcal{O}(bJ)) \equiv \chi(\mathcal{O}, \mathcal{O}(bJ)) + \text{orb}_2(b) + \text{orb}_5(b) \quad (4.15)$$

where $\text{orb}_2(b) = (-)^b/32$ and

$$\text{orb}_5(b) = \begin{cases} -2/25 & \text{for } b = 0, 1, 4 \pmod{5} \\ 3/25 & \text{for } b = 2, 3 \pmod{5} \end{cases}$$

The two terms reflect the \mathbb{Z}_2 and \mathbb{Z}_5 singularities in the weighted projective space $\mathbb{P}^{1,1,1,2,5}$. The modified Euler matrix is found to be

$$I_{ab} = \begin{pmatrix} 1 & 3 & 7 & 13 & 22 & 35 & 53 & 77 & 108 & 147 \\ 0 & 1 & 3 & 7 & 13 & 22 & 35 & 53 & 77 & 108 \\ 0 & 0 & 1 & 3 & 7 & 13 & 22 & 35 & 53 & 77 \\ 0 & 0 & 0 & 1 & 3 & 7 & 13 & 22 & 35 & 53 \\ 0 & 0 & 0 & 0 & 1 & 3 & 7 & 13 & 22 & 35 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 7 & 13 & 22 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 7 & 13 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.16)$$

We now give the exact sequences which define $-S_2$ and S_3

$$0 \rightarrow (-S_2) \rightarrow \mathcal{O}^{\oplus 3} \xrightarrow{f} \mathcal{O}(J) \rightarrow 0 \quad (4.17)$$

where $f^i = \phi_i$ for $i = 1, 2, 3$. The next bundle is given by the exact sequence

$$0 \rightarrow S_3 \rightarrow \mathcal{O}^{\oplus 2} \rightarrow (-S_2) \otimes \mathcal{O}(J) \rightarrow 0 \quad (4.18)$$

The $\sum_a l_a = 0$ bundles are

$$\begin{aligned} \text{ch}(V_1) &= 1 \\ \text{ch}(V_2) &= -2 + J + \frac{J^2}{2} + \frac{J^3}{6} \\ \text{ch}(V_3) &= -J + \frac{J^2}{2} + \frac{5J^3}{6} \\ \text{ch}(V_4) &= 2 - J - \frac{J^2}{2} + \frac{5J^3}{6} \\ \text{ch}(V_5) &= -1 + J - \frac{J^2}{2} + \frac{J^3}{6} \end{aligned} \quad (4.19)$$

Observe that V_3 is a sheaf. The Chern character of sheaves have the pattern $\text{ch}V_{i+5} = -\text{ch}V_i$ and hence relate branes to anti-branes. We thus have given the

Chern character for only the first five sheaves. The corresponding RR charges are

$$\begin{aligned}
\vec{n}_1 &= (-1, 0, 0, 0) \\
\vec{n}_2 &= (2, -1, -1, 3) \\
\vec{n}_3 &= (0, 1, 0, -2) \\
\vec{n}_4 &= (-2, 1, 1, -2) \\
\vec{n}_5 &= (1, -1, 0, 3)
\end{aligned} \tag{4.20}$$

Again, these agree with the corresponding Gepner model results modulo the sign of n_6 [6].

4.3 The two parameter models

The Todd class is defined by

$$\text{Td}(X) = \left(\frac{H}{1 - e^{-H}} \right)^2 \left(\frac{qH}{1 - e^{-qH}} \right) \left(\frac{L}{1 - e^{-L}} \right)^2 \left(\frac{H - 2L}{1 - e^{2L-H}} \right)$$

where $q = 1$ for $\mathbb{P}^{1,1,2,2,2}$ and $q = 3$ for $\mathbb{P}^{1,1,2,2,6}$. One has the intersection relations

$$\begin{aligned}
X = \mathbb{P}^{1,1,2,2,2} : & \quad \langle H^4 \rangle_X = 2 \langle H^3 L \rangle_X = 2, \quad L^2 = 0 \\
X = \mathbb{P}^{1,1,2,2,6} : & \quad \langle H^4 \rangle_X = 2 \langle H^3 L \rangle_X = 2/3, \quad L^2 = 0
\end{aligned}$$

We define h and l (which generate $H_2(M)$) satisfying

$$\begin{aligned}
\langle h \cdot H \rangle_M &= 1 & \langle h \cdot L \rangle_M &= 0 \\
\langle l \cdot H \rangle_M &= 0 & \langle l \cdot L \rangle_M &= 1
\end{aligned}$$

It is easy to work out the relationship between (h, l) and (H^2, HL) . The Chern character of a bundle E is given in terms of the RR-charges of the associated D-brane by (when α_{ab} are set to zero)

$$\begin{aligned}
\text{ch}_0(E) &= -n_6 \\
\text{ch}_1(E) &= -n_4^1 H - n_4^2 L \\
\text{ch}_2(E) &= -n_2^1 h - n_2^2 l \\
\text{ch}_3(E) &= n_0 + 2\beta_1 n_4^1 + 2\beta_2 n_4^2
\end{aligned} \tag{4.21}$$

Both models that we consider are K3 fibrations over a base \mathbb{P}^1 . The K3 fibre in the case of $\mathbb{P}^{1,1,2,2,2}$ [8] is a quartic in in \mathbb{P}^3 while the K3 fibre in the case of $\mathbb{P}^{1,1,2,2,6}$ [12] is a degree six surface in $\mathbb{P}^{1,1,1,3}$. We will see that the helix structure reflects this structure. The helix will can be obtained (loosely-speaking) from the tensor product of two helices

$$\mathcal{R} = (\mathcal{O}, \mathcal{O}(L)) \otimes (\mathcal{O}, \mathcal{O}(H), \dots, \mathcal{O}(dH))$$

where the first helix is associated with the base and the other with the K3 fibre.

4.3.1 $\mathbb{P}^{1,1,2,2,2}[8]$

Under the operations $\theta_i \rightarrow \theta_i + 2\pi$, one obtains $\mathcal{O} \rightarrow \mathcal{O}(H)$ and $\mathcal{O} \rightarrow \mathcal{O}(L)$. Thus, conjecture one predicts that the line bundles $\mathcal{O}(aH + bL)$ should form a helix. By explicitly computing the Euler form, we obtain the following helix of period eight, \mathcal{R} given by

$$\mathcal{R} = [\mathcal{O}, \mathcal{O}(L), \mathcal{O}(H), \mathcal{O}(H + L), \mathcal{O}(2H), \mathcal{O}(2H + L), \mathcal{O}(3H), \mathcal{O}(3H + L)]$$

with its Euler matrix being upper triangular as required.

$$I_{ab} = \chi(R_a, R_b) = \begin{pmatrix} 1 & 2 & 6 & 10 & 20 & 30 & 50 & 70 \\ 0 & 1 & 2 & 6 & 10 & 20 & 30 & 50 \\ 0 & 0 & 1 & 2 & 6 & 10 & 20 & 30 \\ 0 & 0 & 0 & 1 & 2 & 6 & 10 & 20 \\ 0 & 0 & 0 & 0 & 1 & 2 & 6 & 10 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.22)$$

We need to now verify the second conjecture i.e., the $\sum_a l_a = 0$ bundles are given by a sequence of left mutations. The bundle $L_{\mathcal{O}}(\mathcal{O}(L))$ is defined by the exact sequence

$$0 \rightarrow L_{\mathcal{O}}(\mathcal{O}(L)) \rightarrow \mathcal{O}^{\oplus 2} \xrightarrow{J} \mathcal{O}(L) \rightarrow 0 \quad (4.23)$$

where $J^i = \phi_i$ for $i = 1, 2$. The line bundle obtained this way can be seen to equal $\mathcal{O}(-L)$.

The Chern character of the bundles obtained on restriction to the Calabi-Yau hypersurface are given by

$$\begin{aligned} \text{ch}(V_1) &= 1 \\ \text{ch}(V_2) &= -1 + L \\ \text{ch}(V_3) &= -3 + H - 2L + 4h + 2l + \frac{4}{3} \\ \text{ch}(V_4) &= 3 - H - L - 2l + \frac{2}{3} \\ \text{ch}(V_5) &= 3 - 2H + 4L - 8h + \frac{4}{3} \\ \text{ch}(V_6) &= -3 + 2H - L - \frac{4}{3} \\ \text{ch}(V_7) &= -1 + H - 3L + 4h - 2l - \frac{8}{3} \\ \text{ch}(V_8) &= 1 - H + L + 2l + \frac{2}{3} \end{aligned} \quad (4.24)$$

These results agree with the Gepner model results [6] and also agree with the bundles given by Douglas and Diaconescu [11].

4.3.2 $\mathbb{P}^{1,1,2,2,6}[12]$

Under the operations $\theta_i \rightarrow \theta_i + 2\pi$, one obtains $\mathcal{O} \rightarrow \mathcal{O}(H)$ and $\mathcal{O} \rightarrow \mathcal{O}(L)$. Thus, conjecture one predicts that the line bundles $\mathcal{O}(aH + bL)$ should form a helix. By explicitly computing the Euler form, we obtain the following helix of period twelve, \mathcal{R} given by

$$\mathcal{R} = [\mathcal{O}, \mathcal{O}(L), \mathcal{O}(H), \mathcal{O}(H + L), \dots, \mathcal{O}(5H), \mathcal{O}(5H + L)]$$

The ambient weighted projective space X has as \mathbb{Z}_6 singularity and we need to define a modified Euler form. The modified Euler form is given by

$$\widehat{\chi}(R_a, R_b) \equiv \chi(R_a, R_b) + \text{orb}_6(b - a) \quad (4.25)$$

where

$$\text{orb}_6(b) = \begin{cases} -2/27 & \text{for } b = 0, 1, 5 \pmod{6} \\ 1/27 & \text{for } b = 2, 4 \pmod{6} \\ 4/27 & \text{for } b = 3 \pmod{6} \end{cases}$$

$$\widetilde{\chi}(R_a, R_b) = \begin{pmatrix} 1 & 2 & 5 & 8 & 14 & 20 & 31 & 42 & 60 & 78 & 105 & 132 \\ 0 & 1 & 2 & 5 & 8 & 14 & 20 & 31 & 42 & 60 & 78 & 105 \\ 0 & 0 & 1 & 2 & 5 & 8 & 14 & 20 & 31 & 42 & 60 & 78 \\ 0 & 0 & 0 & 1 & 2 & 5 & 8 & 14 & 20 & 31 & 42 & 60 \\ 0 & 0 & 0 & 0 & 1 & 2 & 5 & 8 & 14 & 20 & 31 & 42 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 5 & 8 & 14 & 20 & 31 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 5 & 8 & 14 & 20 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 5 & 8 & 14 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.26)$$

One can then determine the Chern characters associated with the restrictions of the S_i to the Calabi-Yau hypersurface. We obtain

$$\begin{aligned} \text{ch}(V_1) &= 1 \\ \text{ch}(V_2) &= -1 + L \\ \text{ch}(V_3) &= -2 + H - 2L + 2h + l + \frac{2}{3} \\ \text{ch}(V_4) &= 2 - H - l + \frac{1}{3} \\ \text{ch}(V_5) &= 1 - H + 2L - 2h + l + \frac{4}{3} \\ \text{ch}(V_6) &= -1 + H - L - l - \frac{1}{3} \end{aligned} \quad (4.27)$$

Note that $\text{ch}V_i = -\text{ch}V_{i+6}$ and hence we have not listed the other six cases. The Chern character suggest the following identification among various bundles: $V_{i+1} = -V_i \otimes \mathcal{O}(-L)$ for $i = 1, 3, 5$ and $V_5 = \mathcal{O}(2L - H)$. The RR charges associated with the above vector bundles are (in the convention $(n_6, n_4^1, n_4^2, n_0, n_2^1, n_2^2)$)

$$\begin{aligned}
\vec{n}_1 &= (-1, 0, 0, 0, 0, 0) \\
\vec{n}_2 &= (1, 0, -1, 2, 0, 0) \\
\vec{n}_3 &= (2, -1, 2, 1, -2, -1) \\
\vec{n}_4 &= (-2, 1, 0, -4, 0, 1) \\
\vec{n}_5 &= (-1, 1, -2, 1, 2, -1) \\
\vec{n}_6 &= (1, -1, 1, 2, 0, 1)
\end{aligned} \tag{4.28}$$

This agrees with the corresponding numbers obtained in [6] for the corresponding Gepner model.

5 Quivers from Helices and onward

5.1 The Beilinson quiver

Associated with the foundation of a helix \mathcal{R} , made up of a strongly exceptional collection, one can obtain a quiver in the following manner. If $\mathcal{R} = \bigoplus R_i$, and defining $A = \text{Hom}(\mathcal{R}, \mathcal{R})$, we can identify A as the algebra of a quiver with relations. The corresponding quiver diagram may be identified as follows. The i -th vertex is associated with R_i , or more precisely with $\text{Hom}(R_i, R_i)$. Draw $\dim(\text{Hom}(R_i, R_j))$ arrows beginning at vertex i and ending at vertex j . The quiver relations are the obvious ones, namely those given by considering explicitly the maps involved in $\text{Hom}(R_i, R_j)$. In the case of \mathbb{P}^n , where the $R_i = \mathcal{O}(i-1)$, the relations can be obtained by the obvious rule that $\text{Hom}(\mathcal{O}(i), \mathcal{O}(i+1))$ is given by multiplication by the coordinates of \mathbb{P}^n . Thus for homomorphisms from \mathcal{O} to $\mathcal{O}(2)$, we can have maps that are either of the form $\phi_1\phi_2$ or equivalently $\phi_2\phi_1$ and so on with the other coordinates. Thus if links are labelled $X_{i,i+1}^a$, where $a = 1, \dots, (n+1)$ are related to the coordinates and $i = 1, \dots, (n+1)$ labelled the appropriate node in the quiver, we get the relations

$$X_{i,i+1}^a X_{i+1,i+2}^b = X_{i,i+1}^b X_{i+1,i+2}^a \tag{5.1}$$

Note that there is no link between the $(n+1)$ -th node of the quiver and the first node. We refer to this kind of quiver as a generalized Beilinson quiver following the terminology used for the \mathbb{P}^2 case by Douglas et. al. in ref. [10].

The structure of the quiver and its relations become more complicated for the cases of the weighted projective cases and with extra Kähler moduli. However the rules for working them out remain the same. Even for the one Kähler modulus cases we may have links that connect to other than next-nearest-neighbour

vertices. Such links are obviously due to maps using the coordinates of weighted projective space that have higher weight.

As an illustration we give below two quivers. The first one is for the case of \mathbb{P}^2 (figure 1) and the second for the case of $\mathbb{P}^{1,1,1,1,2}$ (figure 2).

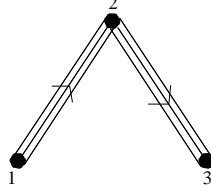


Figure 1: Beilinson quiver for \mathbb{P}^2

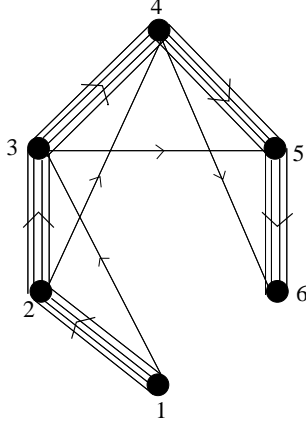


Figure 2: Beilinson quiver for $\mathbb{P}^{1,1,1,1,2}$

A representation of a Beilinson quiver for \mathbb{P}^n is characterised by a dimension vector $(u_1, u_2, \dots, u_{n+1})$ which can be associated to a complex, the Beilinson complex

$$0 \rightarrow \mathbb{C}^{u_1} \otimes \Omega^n(n) \xrightarrow{X_{1,2}} \mathbb{C}^{u_2} \otimes \Omega^{n-1}(n-1) \xrightarrow{X_{2,3}} \mathbb{C}^{u_3} \otimes \Omega^{n-2}(n-2) \rightarrow \dots \\ \dots \rightarrow \mathbb{C}^{u_n} \otimes \Omega^1(1) \rightarrow \mathbb{C}^{u_{n+1}} \otimes \mathcal{O} \rightarrow 0 \quad . \quad (5.2)$$

where $V = \mathbb{C}^{n+1}$ with basis (e_a) . The maps $X_{i,i+1}$ are given by $X_{i,i+1}^a e_a$. This turns the quiver relations into the statement $X_{i,i+1} \cdot X_{i+1,i+2} = 0$ as is to be expected if the sequence above is to be a complex. If the sequence is exact (for instance) at all nodes except one, we can write the Chern character for the sheaf \mathcal{E} corresponding to the appropriate quiver representation as

$$\pm \text{ch}(\mathcal{E}) = \sum_i (-1)^i u_i \text{ch}(\Omega^{n-i}(n-i)) \quad (5.3)$$

Thus we may write characterise any sheaf \mathcal{E} that arises as the cohomology of a complex made up of the $\Omega^i(i)$, by a vector (u_1, \dots, u_{n+1}) . The charges of the corresponding D-brane configuration may be read off using the formula (5.3).

The above complex does not generate all sheaves on \mathbb{P}^n . The more general situation has been considered by Beilinson [30]. He has shown that any sheaf \mathcal{E} on \mathbb{P}^n is obtained from a complex K^\bullet with $H^0(K^\bullet) = \mathcal{E}$, and $H^i(K^\bullet) = 0$ for $i \neq 0$. The i -th term of the complex is isomorphic to $\oplus_j H^{i+j}(\mathcal{E}(-j)) \otimes \Omega^j(j)$. Thus, the Chern character of a generic sheaf (in the large volume limit) is again given by the formula (5.3) with an important difference in that the u_i can be both positive and negative. However from the point of view of the corresponding quiver gauge theory it is clear that we are restricted to only positive values for *all* u_i or negative values for all u_i .

We may suggestively re-write both the complex (5.2) and the corresponding formula for an arbitrary sheaf \mathcal{E} by replacing every $\Omega^i(i)$ by $L^i(\mathcal{O}(i))$ in the notation of Sec. 3 of this paper. Similarly, the i -th term in the Beilinson complex can be rewritten as $\oplus_j H^{i+j}(R_j^* \otimes \mathcal{E}) \otimes S_j$. This is the form in which we expect that the Beilinson complex and Beilinson's theorem to generalize for the more complicated cases under consideration in this paper. We note here that the re-writing of Beilinson's theorem in this language was pointed out by Gorodontsev and Rudakov [31] and also appears for instance in the work of Drézet [32].

The fact that such a generalization is possible is suggested strongly by the work of Bondal [29] [33]. In [29] Bondal shows that bounded derived category of coherent sheaves on a variety X , denoted $\mathcal{D}^b(\text{Coh}(X))$ is equivalent to the bounded derived category of right modules of an algebra A , denoted by $\mathcal{D}^b(\text{mod-}A)$. In fact, in our construction of the generalized Beilinson quiver from the helix, we have already implicitly assumes this categorical equivalence. From Bondal's work, it is clear that the $\{S_i\}$ (called left-dual by him) that we have defined from the $\{R_i\}$ are in fact related to the irreducible representations of the quiver algebra. Note that in formula (5.3) we have explicitly included a sign in the formula for $\text{ch}(\mathcal{E})$ on the right hand side. If we had considered the objects S_i rather than the $\Omega^i(i)$, all the signs would have been fixed to be positive.

We can thus proceed to state the result for the characterisation of all sheaves \mathcal{E} that arises as the cohomology of a complex of sheaves whose elements are the S_i that we have written down in the general case. All such sheaves are characterised by a vector (u_1, \dots, u_n) , where n is fixed suitably by the actual variety under consideration. Further

$$\text{ch}(\mathcal{E}) = \sum_i u_i \text{ch}(S_i) \tag{5.4}$$

In the large-volume limit, the numbers u_i can be positive or negative, whereas at the Gepner point, these numbers are restricted to be positive (or equivalently all negative). We claim that formula (5.4) above is the generalization of formula (5.3) valid for \mathbb{P}^n . Note that this in no way guarantees that the sheaf E is stable or semi-stable. That has to be checked separately as has been noted by [11].

Note that, as pointed out in [9], the requirement that the u_i be positive at the orbifold or Gepner point is a physical one. From all known constructions for orbifolds it is clear that the D-brane spectrum there does not separately produce brane-anti-brane bound states. Any such bound state is already present in the spectrum. Hence there is no need to add anti-branes by taking some of the u_i to be negative as in the theory at large volume. However this is not the case if we move away from the orbifold point.

The dimension of the moduli space of the quiver gauge theory is given by the formula [10]

$$d = 1 - \frac{1}{2}u^t \cdot C \cdot u \quad (5.5)$$

where C is the *Cartan matrix* associated with the quiver. We claim that the natural Euler form associated with the quiver is given by the (modified) Euler form associated with the helix. To be precise, we claim that the Euler form associated with two representations u and v of the Beilinson quiver is given by

$$\langle u, v \rangle = \sum_{ij} u_i v_j \chi(S_i, S_j) \quad (5.6)$$

and the Cartan matrix is given by the symmetrization of the Euler form

$$(u, v) = \langle u, v \rangle + \langle v, u \rangle \quad (5.7)$$

Hence,

$$d = 1 - \frac{1}{2}(u, u) \quad (5.8)$$

We have checked this formula in some simple examples. However, the quiver gauge theory may have several branches and this formula holds for a specific branch⁸.

5.2 Obtaining the McKay quiver

So far we have been restricting our attention to what we have called the generalized Beilinson quiver. However, in general at an orbifold point, and presumably more generally, at a Landau-Ginzburg point, this cannot be the full story. From the work of [10], it is clear that the quiver associated to the orbifold point is the McKay quiver. In the case of \mathbb{P}^n , the McKay quiver is a closed quiver and the Beilinson quiver is produced by truncating one of the links. Thus, in the \mathbb{P}^2 case the McKay quiver is a triangle with three vertices with three lines joining each vertex in ordered fashion. This clearly reflects the Z_3 symmetry of the orbifold point (see figure 3).

⁸We thank M. Douglas for an useful communication in this regard. Details of these issues are to appear in his forthcoming paper.

How can we reconstruct the McKay quiver if we began with the Beilinson quiver? We claim that the answer is clear from the construction of the GLSM for the blow up of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold. While in the large-volume limit, which is the setting for our construction of the R_i , the S_i and the corresponding quiver, we have set the p -field to zero, this is not true at the orbifold point. At the orbifold point, the p -field, which has charge -3 in this case, acquires a non-trivial vacuum expectation value, which gives rise to the twisted sectors of the orbifold theory. The combinations $p\phi_i$, (for $i = 1, 2, 3$) where the ϕ_i are the homogeneous coordinates of \mathbb{P}^2 , in fact can give rise to homomorphisms from $\mathcal{O}(2)$ to \mathcal{O} because the combinations are of degree -2 . This restores the missing link away from the large-volume limit. We can implement this procedure in any of the theories that we have considered. We give below the re-construction of the McKay-type quiver for \mathbb{P}^2 (figure 3) as well as for $\mathbb{P}^{1,1,1,1,2}$ (figure 4). We depict the links obtained by using the p -field by dashed arrows. The quivers for the

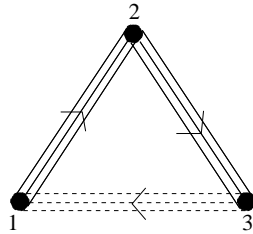


Figure 3: McKay quiver for \mathbb{P}^2

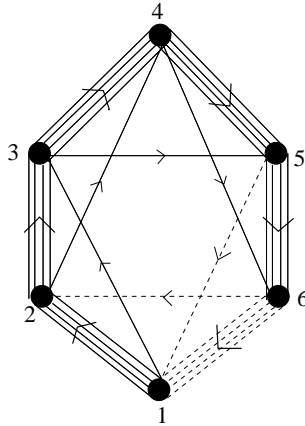


Figure 4: McKay quiver for $\mathbb{P}^{1,1,1,1,2}$

other examples that we consider can be worked out in a similar fashion.

Note that this reconstruction of the McKay quiver from the Beilinson quiver gives us non-trivial information. For instance, bound states that use the link due to the p -field at the orbifold point are allowed in the spectrum but are likely to

decay before we reach the large-volume limit, since there is no such link there and we have only the Beilinson quiver. This point has also been made earlier, in [9], but the connection to the p -field was not realised there.

Another important example which illustrates the role of the p -field is the flop transition as seen by the quiver gauge theory. Let us use the local model for the flop as given in [7] where there are four fields, (ϕ_1, \dots, ϕ_4) , with charges $(1, 1, -1, -1)$. In the limit of large and positive Kähler parameter r , the fields ϕ_3 and ϕ_4 are set to zero. These play the role of p -fields in this limit. The corresponding quiver is given by the two nodes with two arrows joining them as in the first of the diagrams in fig. 5. In the flopped phase, where r is large and negative, ϕ_1 and ϕ_2 are set to zero and the other two are non-zero. The quiver for this phase is given by reversing the two arrows as in the second diagram in fig. 5.

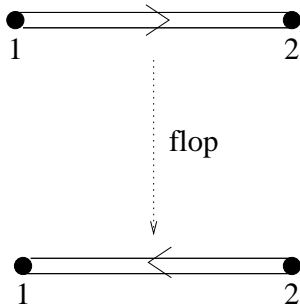


Figure 5: Quivers for the flop transition

It is interesting to note that this operation, in the derived category $\mathcal{D}^b(\text{mod } A)$ is equivalent to a change of t -structure. In general, the addition and removal of arrows, as well as the reversal of the direction of all arrows at a sink, under appropriate circumstances, correspond to changes of t -structure. We will not enter into a detailed discussion of the concept here but refer the reader to [34] for definitions as well as other details.

This procedure, of using the p -fields to add links in the quiver diagram appears to be completely general. In general at different limit points corresponding to different “phases” of the CY manifold, we can re-construct a quiver that would be in part, Beilinson-like, and in part, McKay-like. Given that we can do this from our considerations at the large volume end, we can thus obtain non-trivial information about the D-brane spectrum at different limit points deep inside different “phases” in the Kähler moduli space of CY manifolds. It would be interesting to apply this suggested technique to study the D-brane spectrum at a limit point in the hybrid phase of a two-Kähler modulus example and verify these, if possible, by an explicit computation using the periods of the CY manifold at such a point.

Note however that our procedure is not entirely complete. In general, on a n -dimensional (weighted) projective space considered at the orbifold or Gepner point, there are not only states in the vector representation of $U(n)$ but also states in other, higher-dimensional representations. We have not included these in our considerations so far, neither in the Beilinson quiver nor in the corresponding McKay quiver.

It is also time to sound a note of caution on one issue about which we have been somewhat cavalier so far. All our considerations above apply to constructions in the ambient variety X in which the Calabi-Yau manifold is embedded as a hypersurface M . Thus to recover objects on the CY manifold we need to restrict these structures to the CY manifold. We have been careful about this in the large-volume limit but we have not really explored this issue at the orbifold point. While this has not been a problem with states associated to the $\sum_a l_a = 0$ orbit, this is unlikely to be true for other states which appear as bound states of the exceptional ones. But this takes us to issues, especially those regarding bound states, on which we will not have much to say in this paper.

It is worth emphasizing that the entire discussion here in sec. 5, regarding quiver theories, is somewhat different in spirit from the study of the McKay correspondence. Given an arbitrary orbifold, it is not clear that there is a canonical resolution of the singularities of that orbifold. However, here our orbifold-like points in Kähler moduli space arise as the well-defined limit of a smooth CY manifold and there is no issue at all about which resolution has to be picked at the orbifold end of the story. But given that our aim here is to study D-brane spectra at different points in the Kähler moduli space of a CY manifold, this is good enough.

6 Conclusion

In this paper, we have presented a method of constructing exceptional bundles on weighted projective spaces using the helix naturally associated with the line bundle \mathcal{O} . These techniques generalise in a rather straightforward manner to spaces such as Grassmannians, products of weighted projective spaces. However, in more general situations such as these, the helix associated with \mathcal{O} no longer consists of only line bundles. For example, the foundation of the helix on the Grassmannian $G(2, 4)$ consists of spinor bundles and line bundles [35]. This is presumably related to non-abelian nature of the GLSM associated with the Grassmannian and will be an interesting testing ground for some of conjectures and may also require some modifications before being adapted to more general situations. In this paper, we have considered situations corresponding to CY compactifications where the CY manifold appears as a hypersurface in weighted projective space. There are other situations, such as F-theory compactifications where projective spaces and their blow-ups occur as the base of an elliptic fibration where our

methods may have direct applications.

The special role played by the line bundle \mathcal{O} is very much in line with recent K-theoretic considerations based on Sen's study of non-BPS branes [36]. For example, in IIB compactifications, all BPS branes (as well as some non-BPS branes) can be obtained as bound states of a certain number of D9 branes and anti-branes. Further, D-brane charges in IIB theories are classified by the K-theory group $K^0(X)$ where X is the spacetime on which the string propagates [37]. If we restrict our considerations to the case for $X = M \otimes \mathbb{R}^{3,1}$, where M is a Calabi-Yau threefold and further, we consider only zero-branes in the non-compact spacetime $\mathbb{R}^{3,1}$, then the charges of the particles (in IIA string theory) will be classified by $K^0(M)$. Just as the D9-brane (and its antibrane) generates all other branes in flat spacetime, it is natural to expect the six-brane (wrapping all of M) to generate all lower branes and hence charges. As we have seen, all exceptional branes can be generated by mutations. Other branes must arise as bound states of these exceptional sheaves. This is in line with the claim of [10] that the $\sum_a l_a \neq 0$ arise as bound states of the $\sum_a l_a = 0$ states. These issues will be discussed in a forthcoming publication.

One issue that we have not considered in this paper is the precise location of lines of marginal stability. As we have pointed out in the introduction itself, our analysis allows us to identify candidates for stable objects at special points in the Kähler moduli space. It will be interesting to understand in more precise terms, where and how objects decay. It seems to us that the GLSM would be the right setting to study this issue further. Maybe, this might also be the way to obtain the flow of gradings proposed by [9, 12].

In a companion paper [22], we discuss in detail the explicit construction of the exceptional bundles and some of their bound states in the GLSM with boundary. These involve techniques similar to the constructions which appear in $(0, 2)$ compactifications of the heterotic string though there are subtle differences. We suitably modify this construction to describe $(0, 2)$ multiplets on the boundary of the worldsheet. As in our earlier paper [18], an important issue is the inclusion of an appropriate contact term in order to have a sensible NLSM limit to the GLSM. We find that this construction allows us to describe the tensoring of vector bundles by suitable line bundles under large volume monodromy transformations.

Acknowledgments We thank J. Biswas, K. Paranjape, R. Parthasarathy (Tata Institute) and S. Ramanan for their patience in answering our many queries and illuminating discussions. S.G. is supported in part by the Department of Science and Technology, India under the grant SP/S2/E-03/96.

A Mutations and Diagonalisation of Upper Triangular matrices

Consider a helix $\mathcal{R} = (R_1, \dots, R_n)$ of period n . The intersection matrix $\chi(\mathcal{R}, \mathcal{R})$ is upper-triangular with ones on the diagonal. We shall now prove that the process of left mutations corresponds to the diagonalisation of the upper triangular matrix $I_{ij} \equiv \chi(\mathcal{R}, \mathcal{R})$. We will define the following sets of bundles $\mathcal{R}^{(p)}$, which are generated from the original helix \mathcal{R} . Note that the sets $\mathcal{R}^{(p)}$ for $p \neq 0, (n-1)$ are not helices. Consider

$$\mathcal{R}^{(1)} = (R_1, L_1(R_2), L_2(R_3), \dots, L_{n-1}(R_n))$$

and consider the matrix $I_{ij}^{(1)} \equiv \chi(\mathcal{R}, \mathcal{R}^{(1)})$. On using the formula for the Chern character of the mutated bundle⁹ $\text{ch}(L_{E_i}(E_j)) = \text{ch}(E_j) - \chi(E_i, E_j)\text{ch}(E_i)$, one can see that the matrix $I_{ij}^{(1)}$ is also upper-triangular with ones in the diagonal. In addition, $I_{i,i+1}^{(1)} = 0$. The next set of left mutations will preserve the upper-triangular nature and will have $I_{i,i+1}^{(2)} = I_{i,i+2}^{(2)} = 0$. This is given by

$$\mathcal{R}^{(2)} = (R_1, L_1(R_2), L_1L_2(R_3), \dots, L_{n-2}L_{n-1}(R_n))$$

Notice, that the first two terms of $\mathcal{R}^{(2)}$ and $\mathcal{R}^{(1)}$ are identical and in general, the first $(p-1)$ terms of the collections $\mathcal{R}^{(p)}$ and $\mathcal{R}^{(p-1)}$ coincide. The process ends after $(n-1)$ steps, after which the matrix $I_{ij}^{(n)}$ is the identity matrix.

Thus, the sequence of left mutations gives rise to $\mathcal{S} \equiv \mathcal{R}^{(n)}$ with the property

$$\chi(\mathcal{R}, \mathcal{S}) = \text{the identity matrix}$$

References

- [1] A. Recknagel and V. Schomerus, ‘‘D-branes in Gepner Models,’’ Nucl. Phys. **B531** (1998) 185, [hep-th/9712186](#).
- [2] I. Brunner, M. R. Douglas, A. Lawrence and C. Römelsberger, ‘‘D-branes on the Quintic,’’ JHEP **0008** (2000) 015, [hep-th/9906200](#).
- [3] D.-E. Diaconescu and J. Gomis, ‘‘Fractional Branes and Boundary States in Orbifold Theories, JHEP **0010** (2000) 001, [hep-th/9906242](#).
- [4] M. Gutperle and Y. Satoh, ‘‘D-branes in Gepner models and supersymmetry,’’ Nucl. Phys. **B543** (1999) 73, [hep-th/9808080](#).

⁹We have, for simplicity, assumed that the map $\text{Ext}^0(E_i, E_j) \otimes E_i \rightarrow E_j$ is injective. The other case will include an additional minus sign, which can be compensated for in the definition of the entries in $\mathcal{R}^{(p)}$ and is not important.

- D.-E. Diaconescu and C. Römelsberger, “D-Branes and Bundles on Elliptic Fibration,” Nucl. Phys. **B574** (2000) 245-262, [hep-th/9910172](#).
- P. Kaste, W. Lerche, C.A. Lutken and J. Walcher, “D-branes on K3 - Fibrations,” [hep-th/9912147](#)
- M. Naka and M. Nozaki, “Boundary states in Gepner models,” [hep-th/0001037](#).
- I. Brunner and V. Schomerus, “D-branes at Singular Curves of Calabi-Yau Compactifications,” [hep-th/0001132](#).
- J. Fuchs, C. Schweigert, J. Walcher, “Projections in string theory and boundary states for Gepner models,” [hep-th/0003298](#).
- [5] S. Govindarajan, T. Jayaraman and T. Sarkar, “Worldsheet approaches to D-branes on supersymmetric cycles,” Nucl. Phys. **B580** (2000) 519-547, [hep-th/9907131](#).
- S. Govindarajan and T. Jayaraman, “On the Landau-Ginzburg description of Boundary CFTs and special Lagrangian submanifolds, JHEP **07** (2000) 016, [hep-th/0003242](#).
- [6] E. Scheidegger, “D-branes on some one- and two-parameter Calabi-Yau hypersurfaces,” JHEP **0004** (2000) 003, [hep-th/9912188](#).
- [7] E. Witten, “Phases of $N = 2$ Theories in Two Dimensions,” Nucl. Phys. **B403** (1993) 159, [hep-th/9301042](#).
- [8] M.R. Douglas, “Topics in D-Geometry,” Class. Quant. Grav. **17** (2000) 1057, [hep-th/9910170](#).
- [9] M. R. Douglas, B. Fiol and C. Römelsberger, “Stability and BPS branes,” [hep-th/0002037](#).
- [10] M. R. Douglas, B. Fiol and C. Römelsberger, “The spectrum of a non-compact Calabi-Yau manifold,” [hep-th/0003263](#).
- [11] M. R. Douglas and D.-E. Diaconescu, “D-branes on Stringy Calabi-Yau Manifolds,” [hep-th/0006224](#).
- [12] M. R. Douglas, “D-branes on Calabi-Yau manifolds,” [math.ag/0009209](#).
- [13] M. R. Douglas, B. Greene, D. Morrison, “Orbifold Resolution by D-branes,” Nucl. Phys. **B506** (1997) 84, [hep-th/9704151](#).
- [14] T. Muto, “D-branes on orbifolds and topology change,” Nucl. Phys. **B521** (1998) 183, [hep-th/9711090](#).
- B. Greene, “D-brane topology changing transitions,” Nuc. Phys. **B525** (1998) 284, [hep-th/9711124](#).
- S. Mukhopadhyay and K. Ray, “Conifolds from D-branes,” Phys. Lett. **B423**

- (1998) 247, [hep-th/9711131](#).
- B. Feng, A. Hanany and Y. He, “D-brane gauge theories from toric singularities and toric duality,” [hep-th/0003085](#).
- T. Sarkar, “D-brane gauge theories from toric singularities of the form \mathbb{C}^3/Γ and \mathbb{C}^4/Γ ,” [hep-th/0005166](#).
- [15] Y. Ito and H. Nakajima, “McKay correspondence and Hilbert schemes in dimension three,” [math.AG/9803120](#).
- [16] Miles Reid, “McKay correspondence,” [alg-geom/9702016](#).
- [17] Miles Reid, “La correspondance de McKay,” Séminaire Bourbaki, 52ème année, Novembre 1999, no. 867, [math.AG/9911165](#), to appear in Astérisque 2000.
- [18] S. Govindarajan, T. Jayaraman and T. Sarkar, “On D-branes from Gauged Linear Sigma Models,” [hep-th/0007075](#), to appear in Nuc. Phys. B.
- [19] K. Hori, A. Iqbal and C. Vafa, “D - Branes And Mirror Symmetry,” [hep-th/0005247](#).
- [20] *Helices and Vector Bundles: Seminaire Rudakov*, A.N. Rudakov et. al., Lond. Math. Soc. Lec. Notes Series 148, Cambridge Univ. Press (1990).
- [21] E. Zaslow, “Solitons and Helices: the search for a math-physics bridge,” *Comm. Math. Phys.* **175** (1996) 337, [hep-th/9408133](#).
- [22] S. Govindarajan and T. Jayaraman (to appear)
- [23] B.H. Lian, A. Todorov, S.-T. Yau, “Maximal Unipotent Monodromy for Complete Intersection CY Manifolds,” [math/0008061](#).
- [24] A. Klemm and S. Theisen, “Considerations of one-modulus Calabi-Yau compactifications: Picard-Fuchs equations, Kähler potentials and mirror maps,” *Nucl. Phys.* **B389** (1993) 153-180 [hep-th/9205042](#)
- [25] A. Font, “Periods and Duality Symmetries in Calabi-Yau Compactifications,” *Nucl. Phys.* **B391** (1993) 358, [hep-th/9203084](#).
- [26] P. Candelas, X. de la Ossa, A. Font, S. Katz and D. R. Morrison, “Mirror Symmetry for Two Parameter Models – I,” *Nucl. Phys.* **B416** (1994) 481-538, [hep-th/9308083](#)
P. Candelas, A. Font, S. Katz and D. R. Morrison, “Mirror Symmetry for Two Parameter Models – II,” *Nucl. Phys.* **B429** (1994) 626-674, [hep-th/9403187](#)
S. Hosono, A. Klemm and S. Theisen, “Mirror Symmetry, Mirror Map and Applications to Calabi-Yau Hypersurfaces,” *Commun. Math. Phys.* **167** (1995) 301-350, [hep-th/9308122](#)

- [27] A. Ceresole, R. D’Auria, S. Ferrara, W. Lerche, J. Louis, “Picard-Fuchs Equations and Special Geometry,” *Int. J. Mod. Phys.* **A8** (1993) 79-114, [hep-th/9204035](#).
S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, “Mirror Symmetry, Mirror Map and Applications to Complete Intersection Calabi-Yau Spaces,” *Nucl. Phys.* **B433** (1995) 501-554, [hep-th/9406055](#).
- [28] J. A. Harvey and G. Moore, “On the algebra of BPS states,” *Commun. Math. Phys.* **197** (1998) 489-519, [hep-th/9609017](#).
- [29] A.I. Bondal, “Helixes, Representations of Quivers and Koszul Algebras,” in ref. [20].
- [30] A. A. Beilinson, “Coherent sheaves on \mathbb{P}^n and problems of linear algebra,” *Funct. Anal. Appl.* **12** (1978) 214-216.
- [31] A. L. Gorodentsev and A. N. Rudakov, “Exceptional Bundles on Projective Space,” *Duke Math. J.* **54** (1987) 115-130.
- [32] J.-M. Drézet, “Espaces abstraits de morphismes et mutations,” [alg-geom/9603016](#).
- [33] A. .I. Bondal, “Representation of associative algebras and coherent sheaves,” *Math. USSR-Izv.* **34** (1990) 23-42.
- [34] R. Parthasarathy, “t-structures in the derived category of representations of quivers,” *Proc. Indian Acad. Sci. (Math. Sci.)* **98** (1988) 187-214.
- [35] B. V. Karpov, “A symmetric helix on the Plücker quadric,” in [20].
- [36] A. Sen, “Stable non-BPS states in string theory,” *JHEP* **9806**, 007 (1998) [hep-th/9803194](#).
“Stable non-BPS bound states of BPS D-branes,” *JHEP* **9808**, 010 (1998) [hep-th/9805019](#).
A. Sen, “Tachyon condensation on the brane antibrane system,” *JHEP* **9808**, 012 (1998) [hep-th/9805170](#).
- [37] E. Witten, “D - Branes And K - Theory,” *JHEP* **9812** (1998) 019, [hep-th/9810188](#).