The classification of smooth structures on a homotopy complex projective space

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Abstract

We classify, up to diffeomorphism, all closed smooth manifolds homeomorphic to the complex projective n-space $\mathbb{C}\mathbf{P}^n$, where $n = 3$ and 4. Let M^{2n} be a closed smooth $2n$-manifold homotopy equivalent to $\mathbb{C}\mathbf{P}^n$. We show that, up to diffeomorphism, M^6 has a unique differentiable structure and M^8 has at most two distinct differentiable structures. We also show that, up to concordance, there exist at least two distinct differentiable structures on a finite sheeted cover N^{2n} of $\mathbb{C}\mathbf{P}^n$ for $n = 4, 7$ or 8 and six distinct differentiable structures on N^{10}.

Keywords. complex projective spaces; smooth structures; inertia groups and concordance.

Classification. 57R55; 57R50.

1 Introduction

A piecewise linear homotopy complex projective space M^{2n} is a closed PL $2n$-manifold homotopy equivalent to the complex projective space $\mathbb{C}\mathbf{P}^n$. In [10], Sullivan gave a complete enumeration of the set of PL isomorphism classes of these manifolds as a consequence of his Characteristic Variety theorem and his analysis of the homotopy type of G/PL. He also proved that the group of concordance classes of smoothing of $\mathbb{C}\mathbf{P}^n$ is in one-to-one correspondence with the set of c-oriented diffeomorphism classes of smooth manifolds homeomorphic (or PL-homeomorphic) to $\mathbb{C}\mathbf{P}^n$, where c is the generator of $H^2(\mathbb{C}\mathbf{P}^n;\mathbb{Z})$.

In section 2, we classify up to diffeomorphism all closed smooth manifolds homeomorphic to $\mathbb{C}\mathbf{P}^n$, where $n = 3$ and 4.

Let M^{2n} be a closed smooth $2n$-manifold homotopy equivalent to $\mathbb{C}\mathbf{P}^n$. The surgery theory tells us that there are infinitely many diffeomorphism types in the family of closed smooth

manifolds homotopy equivalent to $\mathbb{C}\mathbf{P}^n$ when $n \geq 3$. In the second section, we also show that if N is a closed smooth manifold homeomorphic to M^{2n}, where $n = 3$ or 4, there is a homotopy sphere $\Sigma \in \Theta_{2n}$ such that N is diffeomorphic to $M\#\Sigma$. In particular, up to diffeomorphism, M^6 has a unique differentiable structure and M^8 has at most two distinct differentiable structures.
In section 3, we prove that if N^{2n} is a finite sheeted cover of $\mathbb{C}\mathbf{P}^n$, then up to concordance, there exist at least $|\Theta_{2n}|$ distinct differentiable structures on N^{2n}, namely $\{[N^{2n}\#\Sigma] \mid \Sigma \in \Theta_{2n}\}$, where $n = 4, 5, 7$ or 8 and $|\Theta_{2n}|$ is the order of Θ_{2n}.

2 Smooth Structures on Complex Projective Spaces

We recall some terminology from [6]:

Definition 2.1. (a) A homotopy m-sphere Σ^m is an oriented smooth closed manifold homotopy equivalent to the standard unit sphere $\mathbb{S}^m$ in $\mathbb{R}^{m+1}$.

(b) A homotopy m-sphere Σ^m is said to be exotic if it is not diffeomorphic to $\mathbb{S}^m$.

(c) Two homotopy m-spheres Σ_1^m and Σ_2^m are said to be equivalent if there exists an orientation preserving diffeomorphism $f : \Sigma_1^m \to \Sigma_2^m$.

The set of equivalence classes of homotopy m-spheres is denoted by Θ_m. The equivalence class of Σ^m is denoted by $[\Sigma^m]$. When $m \geq 5$, Θ_m forms an abelian group with group operation given by connected sum $\#$ and the zero element represented by the equivalence class of $\mathbb{S}^m$. M. Kervaire and J. Milnor [6] showed that each Θ_m is a finite group; in particular, Θ_8 and Θ_{16} are cyclic groups of order 2.

Definition 2.2. Let M be a topological manifold. Let (N, f) be a pair consisting of a smooth manifold N together with a homeomorphism $f : N \to M$. Two such pairs (N_1, f_1) and (N_2, f_2) are concordant provided there exists a diffeomorphism $g : N_1 \to N_2$ such that the composition $f_2 \circ g$ is topologically concordant to f_1, i.e., there exists a homeomorphism $F : N_1 \times [0,1] \to M \times [0,1]$ such that $F_{|N_1 \times 0} = f_1$ and $F_{|N_1 \times 1} = f_2 \circ g$. The set of all such concordance classes is denoted by $\mathcal{C}(M)$.

Start by noting that there is a homeomorphism $h : M^n\#\Sigma^n \to M^n$ $(n \geq 5)$ which is the inclusion map outside of homotopy sphere Σ^n and well defined up to topological concordance. We will denote the class in $\mathcal{C}(M)$ of $(M^n\#\Sigma^n, h)$ by $[M^n\#\Sigma^n]$. (Note that $[M^n\#\mathbb{S}^n]$ is the class of (M^n, Id).)

Theorem 2.3. *(i)* $\mathcal{C}(\mathbb{C}\boldsymbol{P}^3) = 0$.

(ii) $\mathcal{C}(\mathbb{C}\boldsymbol{P}^4) = \{[\mathbb{C}\boldsymbol{P}^4], [\mathbb{C}\boldsymbol{P}^4\#\Sigma^8]\} \cong \mathbb{Z}_2$.

Proof. (i): Consider the following Puppe's exact sequence for the inclusion $i : \mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$ along Top/O:

$$.... \longrightarrow [S\mathbb{CP}^{n-1}, Top/O] \stackrel{(S(g))^*}{\longrightarrow} [\mathbb{S}^{2n}, Top/O] \stackrel{f^*_{\mathbb{CP}^n}}{\longrightarrow} [\mathbb{CP}^n, Top/O] \stackrel{i^*}{\longrightarrow} [\mathbb{CP}^{n-1}, Top/O], \quad (2.1)$$

where $S(g)$ is the suspension of the map $g : \mathbb{S}^{2n-1} \to \mathbb{CP}^{n-1}$. If $n = 2$ or 3 in the above exact sequence (2.1), we can prove that $[\mathbb{CP}^n, Top/O] = 0$. Now by using the identifications $\mathcal{C}(\mathbb{C}\mathbf{P}^3) = [\mathbb{C}\mathbf{P}^3, Top/O]$ given by [7, pp. 194-196], $\mathcal{C}(\mathbb{C}\mathbf{P}^3) = 0$. This proves (i).
(ii): Now consider the case $n = 4$ in the above exact sequence (2.1), we have that $f^*_{\mathbb{C}\mathbf{P}^4} : [\mathbb{S}^8, Top/O] \cong \Theta_8 \mapsto [\mathbb{CP}^4, Top/O]$ is surjective. Then by using [2, Lemma 3.17], $f^*_{\mathbb{C}\mathbf{P}^4}$ is an isomorphism. Hence $\mathcal{C}(\mathbb{C}\mathbf{P}^4) = \{[\mathbb{C}\mathbf{P}^4], [\mathbb{C}\mathbf{P}^4 \# \Sigma^8]\} \cong \mathbb{Z}_2$. This proves (ii). □

Definition 2.4. Let M^m be a closed smooth, oriented m-dimensional manifold. The inertia group $I(M) \subset \Theta_m$ is defined as the set of $\Sigma \in \Theta_m$ for which there exists an orientation preserving diffeomorphism $\phi : M \to M \# \Sigma$.
Define the concordance inertia group $I_c(M)$ to be the set of all $\Sigma \in I(M)$ such that $M \# \Sigma$ is concordant to M.

Theorem 2.5. [3, Theorem 4.2] *For $n \geq 1$, $I_c(\mathbb{CP}^n) = I(\mathbb{CP}^n)$.*

Remark 2.6.

(1) By Theorem 2.3 and Theorem 2.5, $I_c(\mathbb{C}\mathbf{P}^n) = 0 = I(\mathbb{C}\mathbf{P}^n)$, where $n = 3$ and 4.

(2) By Kirby and Siebenmann identifications [7, pp. 194-196], the group $\mathcal{C}(M)$ is a homotopy invariant.

Theorem 2.7. *Let M^{2n} be a closed smooth $2n$-manifold homotopy equivalent to $\mathbb{C}\boldsymbol{P}^n$.*

(i) For $n = 3$, M^{2n} has a unique differentiable structure up to diffeomorphism.

(ii) For $n = 4$, M^{2n} has at most two distinct differentiable structures up to diffeomorphism.

Moreover, if N is a closed smooth manifold homeomorphic to M^{2n}, where $n = 3$ or 4, there is a homotopy sphere $\Sigma \in \Theta_{2n}$ such that N is diffeomorphic to $M \# \Sigma$.

Proof. Let N be a closed smooth manifold homeomorphic to M and let $f : N \to M$ be a homeomorphism. Then (N, f) represents an element in $\mathcal{C}(M)$. By Theorem 2.3 and Remark 2.6(2), there is a homotopy sphere $\Sigma \in \Theta_{2n}$ such that N is concordant to $(M \# \Sigma, Id)$. This implies that N is diffeomorphic to $M \# \Sigma$. This proves the theorem. □

Remark 2.8. Since $\Theta_8 \cong \mathbb{Z}_2$ and $I(\mathbb{C}\mathbf{P}^4) = 0$, by Theorem 2.7, $\mathbb{C}\mathbf{P}^4$ has exactly two distinct differentiable structures up to diffeomorphism.

3 Tangential types of Complex Projective Spaces

Definition 3.1. Let M^n and N^n be closed oriented smooth n-manifolds. We call M a tangential type of N if there is a smooth map $f : M \to N$ such $f^*(TN) = TM$, where TM is the tangent bundle of M.

Example 3.2.

(i) Every finite sheeted cover of $\mathbb{C}\mathbf{P}^n$ is a tangential type of $\mathbb{C}\mathbf{P}^n$.

(ii) Since Borel [1] has constructed closed complex hyperbolic manifolds in every complex dimension $m \geq 1$, by [8, Theorem 5.1], there exists a closed complex hyperbolic manifold M^{2n} which is a tangential type of $\mathbb{C}\mathbf{P}^n$.

Lemma 3.3. [9, Lemma 2.5] *Let M^{2n} be a tangential type of $\mathbb{C}\boldsymbol{P}^n$ and assume $n \geq 4$. Let Σ_1 and Σ_2 be homotopy $2n$-spheres. Suppose that $M^{2n}\#\Sigma_1$ is concordant to $M^{2n}\#\Sigma_2$, then $\mathbb{C}\boldsymbol{P}^n\#\Sigma_1$ is concordant to $\mathbb{C}\boldsymbol{P}^n\#\Sigma_2$.*

Theorem 3.4. [4] *For $n \leq 8$, $I(\mathbb{C}\boldsymbol{P}^n) = 0$.*

Theorem 3.5. *Let M^{2n} be a tangential type of $\mathbb{C}\boldsymbol{P}^n$. Then*

(i) For $n \leq 8$, the concordance inertia group $I_c(M^{2n}) = 0$.

(ii) For $n = 4k+1$, where $k \geq 1$,

$$I_c(M^{2n}) \neq \Theta_{2n}.$$

Moreover, if M^{2n} is simply connected, then

$$I(M^{2n}) \neq \Theta_{2n}.$$

Proof. (i): By Theorem 3.4, for $n \leq 8$, $I(\mathbb{C}\mathbf{P}^n) = 0$ and hence $I_c(\mathbb{C}\mathbf{P}^n) = 0$. Now by Theorem 3.3, $I_c(M^{2n}) = 0$. This proves (i).
(ii): By [5, Proposition 9.2], for $n = 4k+1$, there exists a homotopy $2n$-sphere Σ not bounding spin-manifold such that $\mathbb{C}\mathbf{P}^n\#\Sigma$ is not concordant to $\mathbb{C}\mathbf{P}^n$. Hence by Theorem 3.3,

$$I_c(M^{2n}) \neq \Theta_{2n}.$$

Moreover, $\mathbb{C}\mathbf{P}^n$ is a spin manifold and hence the Stiefel-Whitney class $w_i(\mathbb{C}\mathbf{P}^n) = 0$, where $i = 1$ and 2. Since M^{2n} is a tangential type of $\mathbb{C}\mathbf{P}^n$, there is a smooth map $f : M^{2n} \to \mathbb{C}\mathbf{P}^n$ such that $f^*(T\mathbb{C}\mathbf{P}^n) = TM^{2n}$. This implies that $w_i(M^{2n}) = f^*(w_i(\mathbb{C}\mathbf{P}^n)) = 0$. So, M^{2n} is a spin manifold. If M^{2n} is simply connected, then by [5, Lemma 9.1], $\Sigma \notin I(M^{2n})$ and hence

$$I(M^{2n}) \neq \Theta_{2n}.$$

This proves the theorem. □

Remark 3.6. Let M^{2n} be a tangential type of $\mathbb{C}\mathbf{P}^n$. By Theorem 3.5, up to concordance, there exist at least $|\Theta_{2n}|$ distinct differentiable structures, namely $\{[M^{2n}\#\Sigma] \mid \Sigma \in \Theta_{2n}\}$, where $n = 4, 5, 7$ or 8 and $|\Theta_{2n}|$ is the order of Θ_{2n}.

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